Proofs of character formulas via sheaves on Bruhat graphs

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5. From parity sheaves to multiplicities

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To any $\lambda \in \mathfrak{h}^\star = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ we associate the simple module $L(\lambda)$ with highest weight λ. We want to calculate its *character*

$$\mathrm{char}\ L(\lambda) = \sum_{\mu \in \mathfrak{h}^\star} \dim_{\mathbb{C}} L(\lambda)_\mu e^\mu.$$

Here, $L(\lambda)_\mu$ denotes the μ-eigenspace of the $\mathfrak{h}$-action.

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As an exercise, check that this gives Weyl's character formula in the case $w = e$.

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Again, we want to calculate the character of $L_k(\lambda)$, i.e.

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$$\mathrm{char}\ L_k(\lambda) = \sum_{\mu \in X} \dim_k L_k(\lambda)_\mu e^\mu,$$

where $L_k(\lambda)_\mu$ denotes the μ-eigenspace of the T-action. This is called a *modular character*.

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We denote by h the Coxeter number, i.e. the height of the highest root in R $+1$. Set

$$\widehat{\mathcal{W}}^{res,+} = \{w \in \widehat{\mathcal{W}} \mid 0 \leq \langle w.0, \alpha^\vee \rangle < p \text{ for all simple roots } \alpha\},$$
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$$\widehat{\mathcal{W}}^\circ = \{w \in \widehat{\mathcal{W}} \mid w \leq \widehat{w}_0\},$$
$$\widehat{\mathcal{W}}^{\circ,+} = \{w \in \widehat{\mathcal{W}}^\circ \mid 0 < \langle w.0 + \rho, \alpha^\vee \rangle \text{ for all } \alpha \in \Pi\}.$$

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For any $\lambda \in X^+$ there is a *Weyl module* $H^0(\lambda)$. Its character is given by Weyl's character formula

$$\operatorname{char} H^0(\lambda) = \chi(\lambda) := \frac{\sum_{y \in \mathcal{W}} (-1)^{l(y)} e^{y(\lambda+\rho)}}{\sum_{y \in \mathcal{W}} (-1)^{l(y)} e^{y(\rho)}}.$$

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Conjecture

Suppose that $p > h$. Then for $w \in \widehat{\mathcal{W}}^{res,+}$ we have

$$\text{char } L(w.\lambda) = \sum_{x \in \widehat{\mathcal{W}}^{\circ,+}} (-1)^{l(w)-l(x)} P_{w_0 x, w_0 w}(1) \, \text{char } H^0(x.\lambda).$$

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This conjecture is equivalent to the Kazhdan–Lusztig conjecture.

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Theorem (BGG-reciprocity)

For any $\lambda, \mu \in \mathfrak{h}^\star$ we have

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)].$$

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Let us denote by $\mathcal{O}_{[\lambda]}$ the block of $\mathcal{O}$ that contains $L(\lambda)$. For any simple reflection s there is a *translation functor*

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Observation

If λ is dominant and regular, and if $s \cdots t$ is a reduced expression in $\mathcal{W}$, then

$$\vartheta_t \cdots \vartheta_s P(\lambda) = P(s \cdots t.\lambda) \oplus \bigoplus_{x < s \cdots t} P(x.\lambda)^{\oplus a^x_{s,\ldots,t}}$$

for some numbers $a^x_{s,\ldots,t}$.

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Theorem (Humphreys)

For any $\lambda, \mu \in X$ we have

$$(Q(\lambda) : Z(\mu)) = [Z(\mu) : L'(\lambda)].$$

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It is now convenient to identify $\widehat{\mathcal{W}}$ with the set $\mathcal{A}$ of alcoves via the map $w \mapsto w(A_e)$, where A_e is the unique alcove in the dominant chamber that contains 0 in its closure. Let $A \mapsto w_A$ be the inverse map. We abbreviate

$$Z(A) := Z(w_A.0)$$
$$Q(A) := Q(w_A.0)$$

Then Lusztig's conjecture translates to

$$(Q(A) : Z(B)) = p_{B,A}(1)$$

for all $A, B \in \mathcal{A}$.

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For any simple reflection s there is a translation functor $\theta_s \colon \mathcal{C} \to \mathcal{C}$. It is not difficult to show that

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If $w_0 s \cdots t$ is a reduced expression in $\widehat{\mathcal{W}}$, then

$$\theta_t \cdots \theta_s Q(A_{w_0}) = Q(A_{w_0 s \cdots t}) \oplus \bigoplus_B Q(B)^{\oplus b^B_{s,\ldots,t}}$$

for some numbers $b^B_{s,\ldots,t}$.

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Theorem (Kazhdan–Lusztig)

For any $w \in \mathcal{W}$ there is a unique element $C'_w \in \mathcal{H}$ with

$$\overline{C'_w} = C'_w, \quad C_w \in T_w + \sum_{x \in \mathcal{W}} v\mathbb{Z}[v] T_x.$$

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Observation

Let $w = s \cdots t$ be a reduced expression. Then

$$C'_s \cdots C'_t = C'_w + \sum_{x \leq s \cdots t} \tilde{a}^x_{(s,\dots,t)} C'_x$$

for some numbers $\tilde{a}^x_{(s,\dots,t)}$.

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Observation

For a reduced expression $w_0 s \cdots t$ we have

$$Q_{A_{w_0}} \cdot C'_s \cdots C'_t = Q_{A_{w_0} s \cdots t} + \sum_B \tilde{b}^B_{s,\ldots,t} Q_B$$

for some numbers $\tilde{b}^B_{s,\ldots,t}$.

Decomposition conjecture

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Fact 1

The Kazhdan–Lusztig conjecture is equivalent to

$$a^{x}_{s,\dots,t} = \tilde{a}^{x}_{s,\dots,t}$$

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Fact 2

Lusztig's conjecture is equivalent to

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For a simple reflection s denote by $\pi_s \colon \mathcal{F}\mathrm{l}^\vee \to \mathcal{F}\mathrm{l}^{\vee,s} = G^\vee/P^{\vee,s}$ the canonical map to the *partial* flag variety.

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Let $s \cdots t$ be a reduced expression. Then

$$\pi_t^* \pi_{t*} \cdots \pi_s^* \pi_{s*} P_{k,e} = P_{k,s\cdots t} \bigoplus_{x < s\cdots t} P_{k,x}^{\oplus c^x_{s,\dots,t}}$$

for some numbers $c^x_{s,\dots,t}$.

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Theorem

For any $w \in \mathcal{W}$, $P_{\mathbb{C},w}$ is the $T^\vee$-equivariant intersection cohomology complex on X_w. Moreover, if $s \cdots t$ is a reduced expression, then

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Hence, in order to prove the Kazhdan–Lusztig conjecture one has to look for a connection between parity sheaves and representation theory.

Sheaves on moment graphs

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- a functor $\mathbb{V}$ from a *deformed* version of category $\mathcal{O}$ to sheaves on the Bruhat graph.

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We will prove the following:

Theorem

For any $w \in \mathcal{W}$ we have $\mathbb{V}(P(w.\lambda)) \cong \mathscr{B}(w) \cong \mathbb{W}(P_{\mathbb{C},w})$ and

$$(P(w.\lambda) : M(x.\lambda)) = \operatorname{rk} \mathscr{B}(w)^x = \operatorname{rk} \mathbb{H}^\bullet_{T^\vee}((P_{\mathbb{C},w})_x).$$

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This implies the Kazhdan–Lusztig conjecture.

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We now consider the complex affine flag variety $\widehat{\mathcal{F}\mathrm{l}}^{\vee}$. Denote by $\widehat{T}^{\vee} = T^{\vee} \times \mathbb{C}^{\times}$ the extended (dual) torus. To any $w \in \widehat{\mathcal{W}}$ we associate

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Theorem (AJS)

There is a functor $\mathbb{V}$ from the category of deformed $\mathfrak{g}_k$-T*-modules to $\mathcal{K}$. It satisfies*

$$(Q(A) : Z(B)) = \dim_Q \left(\mathbb{V}Q(A)\right)^B.$$

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Theorem

Suppose that $w \in \mathcal{W}^{\text{res},-}$. There is a functor Φ form sheaves on $\mathcal{G}$ with k-coefficients to $\mathcal{K}$ with $\Phi(\mathscr{B}(w)) \cong \mathbb{V}(Q(A_w))$ and

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Corollary

For $w \in \widehat{\mathcal{W}}^{\text{res},-}$ and char $k > p$ we have

$$\operatorname{rk} \mathbb{H}^\bullet_{T^\vee}((P_{k,w})_x) = (Q(A_w) : Z(A_x)).$$

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Research Problem

Calculate $\operatorname{rk} \mathbb{H}^{\bullet}_{T^\vee}((P_{k,w})_x)$ for char $k > 0$ and prove Lusztig's conjecture.