Moment graphs

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A short history

1998. Moment graphs were introduced by Goresky, Kottwitz and MacPherson as a combinatorial model for the topology of a complex equivariantly formal variety.

2001. Braden and MacPherson gave a description of the intersection cohomologies on suitably stratified complex projective varieties with an equivariantly formal torus action in terms of a canonical sheaf on the associated moment graph.

1998-2001. Guillemin and Zara considered similar structures (in the case of smooth varieties).

A short history

2006-2008. Fiebig used moment graphs and corresponding canonical sheaves to give a new approach to Kazhdan-Lusztig's and Lusztig's conjectures.

2009. Fiebig and Williamson obtained the Braden-MacPherson sheaves by localizing parity sheaves.

Basic definition

Let $Y = \mathbb{Z}^r$ be a lattice.

Definition

A **moment graph** $\mathcal{G}$ *over* Y *is a directed graph* $(\mathcal{V}, \mathcal{E})$ *with the set of vertices* $\mathcal{V}$ *and the set of edges* $\mathcal{E}$ *together with a map* $\alpha : \mathcal{E} \to Y \setminus \{0\}$ *(called the labelling). We assume moreover that a moment graph has neither cycles nor multiple edges.*

Example

To a complex variety with a torus action that has finitely many 0- and 1-dimensional orbits and carries a suitable stratification one associates a moment graph over the character lattice of the torus.

It encodes the structure of the orbits of dimension zero and one. In a later talk we will see a more particular example of a flag variety.

Bruhat graphs

The most important examples of moment graphs for us are the ones associated to a finite or affine root system, we call them **Bruhat graphs**.
Let V be a finite dimensional rational vector space, V^* its dual and $R \subset V$ be a root system.
Let $R^+ \subset R$ the set of positive roots and $W \subset Gl(V)$ the Weyl group corresponding to R.
We denote by

$$X := \{\gamma \in V \mid \langle \gamma, \alpha^\vee \rangle \in \mathbb{Z} \, \forall \alpha \in R\}$$

the weight lattice.

Bruhat graphs

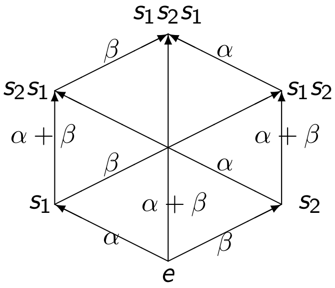
Definition

The moment graph $\mathcal{G}_R$ over the lattice X is given by

- *The set of vertices is W.*
- *Two vertices $x, y \in W$ are connected by an edge E if there is a positive root β such that $x = s_\beta y$. We set $\alpha(E) := \beta$.*
- *An edge connecting x and y is directed towards y if $l(y) > l(x)$, where l is the length function.*

Example.

Let us look at the A_2 root system. There are two simple roots α, β. The Weyl group is generated by the two simple reflections $s_1 = s_\alpha$ and $s_2 = s_\beta$. The Bruhat graph looks like this:



Bruhat graphs

The construction for the affine case is very similar:

- The set of vertices is $\widehat{W}$.
- Two vertices $x, y \in \widehat{W}$ are connected by an edge E if there is a positive affine root β_n such that $x = s_{\beta_n} y$. We set $\alpha(E) := \beta_n$.
- An edge connecting x and y is directed towards y if $l(y) > l(x)$, where l is the length function.

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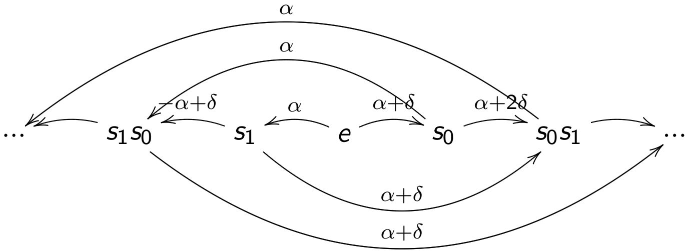
Example

Let's consider now the affine case $\tilde{A}_1$. The positive affine roots are $R^+ = \{\alpha + n\delta \mid n \geq 0\} \cup \{-\alpha + n\delta \mid n > 0\}$, where α is the positive simple root for A_1.

The Bruhat graph looks like this:



GKM-property

Let $\mathcal{G}$ be a moment graph over Y and k a field. Denote by $Y_k = Y \otimes_{\mathbb{Z}} k$ the vector space over k spanned by Y.

Definition

The pair $(\mathcal{G}, k)$ is a *GKM-pair* if $char(k) \neq 2$ and for any vertex x of $\mathcal{G}$ and any two distinct edges E, E' that contain x, the labels $\alpha(E)$, $\alpha(E')$ are linearly independent in Y_k.

Exercise. Let $\mathcal{G}$ be the moment graph associated to A_n root system. Then the pair $(\mathcal{G}, k)$ is a GKM-pair for any field k with $char(k) \neq 2$.

Sheaves on moment graphs

Let $\mathcal{G}$ be a moment graph over a lattice Y.

Let $S = S_k(Y_k)$ be the symmetric algebra of Y_k. This is a polynomial algebra of rank$= dim_k(Y_k)$.

Definition

A k-sheaf $\mathcal{F} = (\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$ on $\mathcal{G}$ is given by the following data:

- an S-module $\mathcal{F}^x$ for any vertex x.
- an S-module $\mathcal{F}^E$ such that $\alpha(E)\mathcal{F}^E = 0$ for each edge E.
- for each edge E with vertex x a homomorphism $\rho_{x,E} : \mathcal{F}^x \to \mathcal{F}^E$ of S-modules.

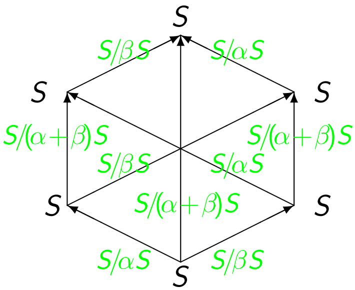
The structure sheaf

The most natural sheaf on a moment graph $\mathcal{G}$ is its structure sheaf $\mathcal{Z}$. It is defined in the following way:

- $\mathcal{Z}^x = S$ for any vertex x,
- $\mathcal{Z}^E = S/\alpha(E)S$ for each edge E,
- for each edge E with vertex x, $\rho_{x,E} : \mathcal{Z}^x \to \mathcal{Z}^E$ is just the canonical projection.

Example

Consider again the A_2 case. The structure sheaf of the associated moment graph is:



Sections of sheaves

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a moment graph over a lattice Y and $\mathcal{F}$ a sheaf on $\mathcal{G}$.

Notation. Let $\mathcal{I} \subset \mathcal{V}$, we denote by $\mathcal{I}$ the complete subgraph of $\mathcal{G}$ having $\mathcal{I}$ as set of vertices as well.

Definition

The **set of sections** *of $\mathcal{F}$ over $\mathcal{I}$ is:*

$$\Gamma(\mathcal{I}, \mathcal{F}) = \left\{ (m_x) \in \prod_{x \in \mathcal{I}} \mathcal{F}^x \mid \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \, \forall E : x \to y \right\}.$$

Notation. We denote by $\Gamma(\mathcal{F}) := \Gamma(\mathcal{V}, \mathcal{F})$ the set of global sections.

Structure algebra

Let $\mathcal{G}$ be a moment graph.

Definition

We call **structure algebra** *of $\mathcal{G}$ the set of global sections of the structure sheaf of $\mathcal{G}$ and we denote it by $Z := \Gamma(\mathcal{G}, \mathcal{Z})$.*

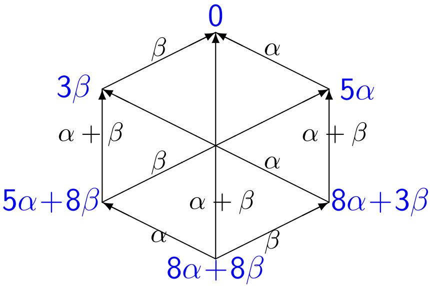
Observe that

- componentwise addition and multiplication give Z the structure of an S-algebra,
- componentwise addition and multiplication give $\Gamma(\mathcal{I}, \mathcal{F})$ a Z-module structure.

Example

Consider the moment graph associated to the A_2 root system.

Let's build an element of its structure algebra:



The Alexandrov topology

We have not used yet the partial order on the set of vertices!

We will use it to define the **Alexandrov topology** on $\mathcal{G}$.

Definition

We say that a subset $\mathcal{I}$ of $\mathcal{V}$ is **open** if and only if it is downwardly closed, i.e. if and only if

$$x \in \mathcal{I}, y \geq x \;\Rightarrow\; y \in \mathcal{I}.$$

The Alexandrov topology

Notation. We denote

$$\{> x\} := \{y \in \mathcal{V} \mid y > x\} \quad \{\geq x\} := \{y \in \mathcal{V} \mid y \geq x\}$$

$$\{< x\} := \{y \in \mathcal{V} \mid y < x\} \quad \{\leq x\} := \{y \in \mathcal{V} \mid y \leq x\}$$

$$[x, z] := \{\geq x\} \cap \{\leq z\} \quad (x, z) := \{> x\} \cap \{< z\}$$

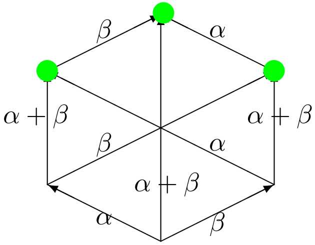
$$(x, z] := \{> x\} \cap \{\leq z\} \quad [x, z) := \{\geq x\} \cap \{< z\}$$

Thus a basis of open sets for the topology defined above is

$$\{\{\geq x\} \mid x \in \mathcal{V}\}.$$

Example

Consider again the moment graph associated to the A_2 root system.



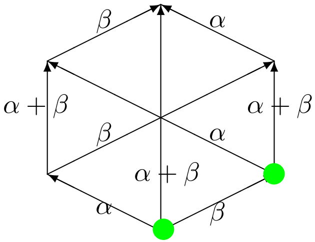
The set

$$\{\ s_1 s_2 s_1,\ s_2 s_1\ ,\ s_1 s_2\ \}$$

is OPEN

Example

Consider again the moment graph associated to the A_2 root system.



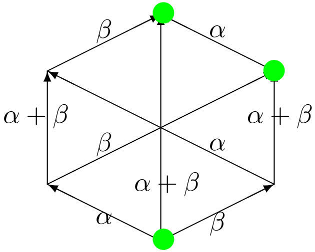
The set

$$\{ s_2 \ , \ e \ \}$$

is CLOSED

Example

Consider again the moment graph associated to the A_2 root system.



The set

$$\{ s_1 s_2 s_1,\ s_2 s_1,\ e \}$$

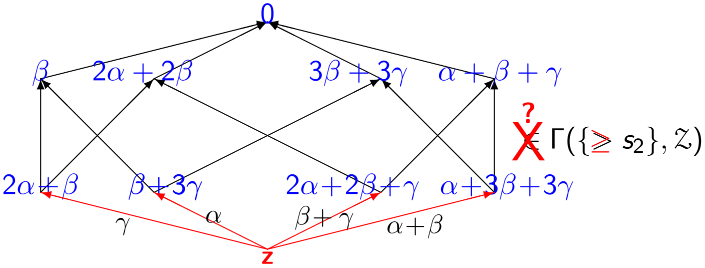
is neither OPEN nor CLOSED

Flabby Sheaves

Problem. Understand whether a sheaf $\mathcal{F}$ on $\mathcal{G}$ is flabby, i.e. whether for any open subset $\mathcal{I} \subseteq \mathcal{V}$ the restriction map $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{I}, \mathcal{F})$ is surjective.

Example

Consider the A_3 root system, having as set of simple roots $\{\alpha, \beta, \gamma\}$. We denote by $s_1 := s_\alpha$, $s_2 := s_\beta$, $s_3 = s_\gamma$ the corresponding reflections. Consider the complete subgraph of $\mathcal{G}_{A_3}$ having as set of vertices the interval $[s_2, s_2 s_1 s_3 s_2]$.



$\mathbf{z} = 2\alpha + \beta + x_1\gamma = (1 + x_4)\alpha + (3 + x_4)\beta + 3\gamma!$

Flabbiness conditions

Let $\mathcal{F}$ be a sheaf on a moment graph $\mathcal{G}$ and $x \in \mathcal{V}$.

Notation. Denote by

- $\mathcal{E}_{\delta x} := \{E \in \mathcal{E} \mid E : x \to y\}$,
- $\mathcal{V}_{\delta x} := \{y \in \mathcal{V} \mid \exists E \in \mathcal{E}_{\delta x} \text{ such that } E : x \to y\}$,
- $\rho_{\delta_x} := (\rho_{x,E})^T_{E \in \mathcal{E}_{\delta x}}$,
- $\mathcal{F}^{\delta x}$: the image of $\Gamma(\{> x\}) := \Gamma(\{> x\}, \mathcal{F})$ under the composition u_x of the following functions:

$$u_x : \Gamma(\{> x\}) \hookrightarrow \prod_{\substack{y \in \mathcal{V} \\ y > x}} \mathcal{F}^y \to \prod_{y \in \mathcal{V}_{\delta x}} \mathcal{F}^y \xrightarrow{\oplus \rho_{y,E}} \prod_{E \in \mathcal{E}_{\delta x}} \mathcal{F}^E$$

.

Flabbiness conditions

Lemma

Let $\mathcal{F}$ be a sheaf on $\mathcal{G}$. The following sentences are equivalent:

(i) *For any open set $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{V}$ the restriction map $\Gamma(\mathcal{J}, \mathcal{F}) \to \Gamma(\mathcal{I}, \mathcal{F})$ is surjective.*

(ii) *For any vertex $x \in \mathcal{V}$, the restriction map $\Gamma(\{\geq x\}, \mathcal{F}) \to \Gamma(\{> x\}, \mathcal{F})$ is surjective .*

(iii) *For any vertex $x \in \mathcal{V}$, the map $\rho_{\delta x} : \mathcal{F}^x \to \prod_{E \in \mathcal{E}_{\delta x}} \mathcal{F}^E$ contains $\mathcal{F}^{\delta x}$ in its image.*

This Lemma motivates the construction of the main object associated to a moment graph.

The Braden-MacPherson Sheaf

Theorem (Braden-MacPherson, 2001)

Let $\mathcal{G}$ be a finite moment graph over Y with highest vertex w. There exists exactly one (up to isomorphism) indecomposable sheaf $\mathcal{B}_w$ on $\mathcal{G}$ with the following properties:

(i) $\mathcal{B}_w^w \cong S$;

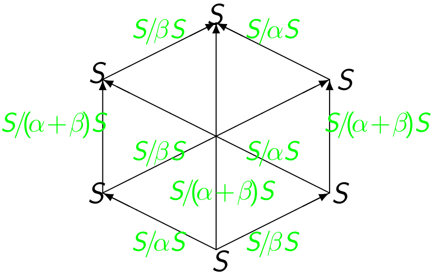
(ii) *If* $x, y \in \mathcal{V}$, $E : x \to y \in \mathcal{E}$, *then* $\mathcal{B}_w^E \cong \mathcal{B}_w^y / \alpha(E) \mathcal{B}_w^y$;

(iii) *If* $x, y \in \mathcal{V}$, $E : x \to y \in \mathcal{E}$, *then* $\rho_{\delta x} : \mathcal{B}_w^x \to \mathcal{B}_w^{\delta x}$ *is a projective cover in the category of graded* S*-modules.*

We call $\mathcal{B}_w$ **Braden-MacPherson** sheaf or **canonical** sheaf.

Example

The *BMP*-sheaf on $\mathcal{G}_{A_2}$ is the following one:

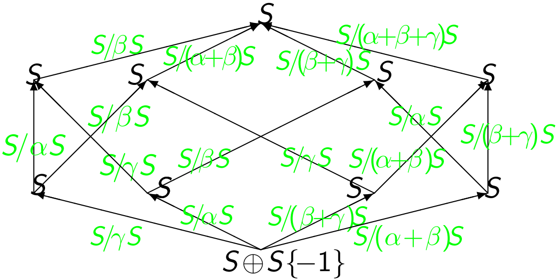


Remark

Any moment graph associated to a finite root system has BMP-sheaf isomorphic to its structure sheaf. This is because the corresponding Schubert variety is smooth.

Example

Consider now the moment graph associated to the interval $[s_2, s_2 s_1 s_3 s_2] \in W$, the Weyl group corresponding to the A_3 root system. It has the following *BMP*-sheaf :



The multiplicity conjecture

Consider a moment graph $\mathcal{G}$ associated to a root system (finite or affine) having Weyl group W.

Conjecture

Le k be a field and $(\mathcal{G}, k)$ a GKM-pair. For every $w \in W$ and for all $x \leq w$ we have:

$$rk\mathcal{B}_w^x = P_{x,w}(1).$$

$P_{x,w}$ are the Kazhdan-Lusztig polynomials.

The multiplicity conjecture in short

The multiplicity conjecture has very important consequences in representation theory.

It is proved if

- $char\ k = 0$
- $\mathcal{B}_w^x \cong S$ if and only if $P_{x,w}(1) = 1$. (Fiebig, 2006)

In positive characteristic the multiplicity conjecture is proven only for p bigger than a huge number, depending on the Weyl group (Fiebig, 2007).