Introduction to Category $\mathcal{O}$

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Motivation

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Classify unitary representations of reductive Lie Groups (e.g. $GL(n)$, $SL(n)$)

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Admissible representations of complex semisimple Lie groups
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Toy model

Category $\mathcal{O}$
(Also includes finite dim and Verma modules)

Definition of Category $\mathcal{O}$

$\mathfrak{g}$ - Semisimple Lie Algebra, $U(\mathfrak{g})$ - its universal enveloping algebra.

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and $U(\mathfrak{g}) = U(\mathfrak{n}_-)U(\mathfrak{h})U(\mathfrak{n}_+)$ (PBW theorem)

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Definition

Category $\mathcal{O}$ is the full subcategory of Mod$U(\mathfrak{g})$ whose objects satisfy the following properties:

- ($\mathcal{O}$1) M is finitely generated
- ($\mathcal{O}$2) M is $\mathfrak{h}$-semisimple, i.e. $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$
- ($\mathcal{O}$3) M is locally $\mathfrak{n}_+$-finite i.e. $\forall v \in M$: $\dim U(\mathfrak{n}_+)v < \infty$

$\mathcal{O}(2) + \mathcal{O}(3) \Rightarrow$ for every $v \in M$ there exists k s.t. $(\mathfrak{n}_+)^k v = 0$

Basic properties

Theorem

Category $\mathcal{O}$ satisfies the following properties:

1. *$\mathcal{O}$ is an Abelian category.*
2. *$\mathcal{O}$ is Noetherian and Artinian.*
3. *$\mathcal{O}$ is closed under submodules,quotients and finite direct sums.*

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3. *$\mathcal{O}$ is closed under submodules,quotients and finite direct sums.*
4. *$\forall M \in \mathcal{O}$ all of the weight spaces M_λ are finite dimensional.*
5. *$\forall M \in \mathcal{O}$ the weights of M are contained in the union of finitely many sets of the form $\lambda - \Gamma$ with $\lambda \in \mathfrak{h}^*$ and Γ the semigroup generated by the positive roots.*
6. *$\forall M \in \mathcal{O}$: M is finitely generated as a $U(\mathfrak{n}_-)$ module.*

Highest weight modules

Definition

Let M be a $U(\mathfrak{g})$ module, then $v^+ \in M$ is a highest weight vector of weight $\lambda \in \mathfrak{h}^*$ if $v^+ \in M_\lambda$ and $\mathfrak{n}_+ v^+ = 0$

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Definition

A $U(\mathfrak{g})$ module M is a heighest weight module of weight λ if there is a highest weight vector $v^+ \in M_\lambda$ s.t. $M = U(\mathfrak{g})v^+$

Highest weight modules

Let M be a heighest weight module of weight λ generated by a maximal vector v^+. Fix an ordering of the positive roots $\alpha_1, \ldots, \alpha_m$ and choose corresponding root vectors $y_i \in \mathfrak{g}_{-\alpha_i}$. Then:

1. M is spanned by the vectors $y_1^{i_1} \cdots y_m^{i_m} v^+$ with $i_j \in \mathbb{Z}^+$, having respective weights $\lambda - \sum i_j \alpha_j$.
2. All weights μ of M satisfy $\mu \leq \lambda$ (i.e. $\mu = \lambda -$ (sum of positive roots), or $\mu \in \lambda - \Gamma$).
3. For all weights μ of M we have dim $M_\mu < \infty$, while dim $M_\lambda = 1$. So M is a weight module, locally $\mathfrak{n}_+$ finite and $M \in \mathcal{O}$.
4. M has a unique maximal submodule and unique simple quotient, in particualr M is indecomposable.

Verma modules

Let $\mathfrak{b} \in \mathfrak{g}$ the Borel subalgebra. Then $\mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$. Let $\mathbb{C}_\lambda$, $\lambda \in \mathfrak{h}^*$ be a 1-dimensional $\mathfrak{b}$ module on which $\mathfrak{n}$ acts trivially and $\mathfrak{h}$ acts by λ.

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Remark

$M(\lambda)$ is a universal heighest weight module of weight λ: For any heighest weight module M of weight λ we have a natural map from $M(\lambda)$ onto M

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Every simple module in $\mathcal{O}$ is isomorphic to some $L(\lambda)$ with $\lambda \in \mathfrak{h}^*$ and is therefore uniquely determined up to isomorphism by its highest weight. Moreover, $\dim Hom_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\mu\lambda}$

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Integral weight lattice: $\Lambda := \{\lambda \in \Phi : \forall \alpha \in \Phi :< \lambda, \alpha^\vee >\in \mathbb{Z}\}$

Theorem

$L(\lambda)$ is finite dimensional iff $\lambda \in \Lambda^+$.
Additionally, in this case $\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu}$ *for any $\mu \in \mathfrak{h}^*$ and $w \in W$.*

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$Z(\mathfrak{g})$ acts on the weight spaces of any $M \in \mathcal{O}$.
In particular, if $M = M(\lambda)$ is a Verma module with weight λ and $v^+ \in M_\lambda$ is the highest weight vector with weight λ then $\forall z \in Z(\mathfrak{g}) : zv^+ = \chi_\lambda(z)v^+$.
χ_λ is called the central character of $M(\lambda)$. Since $M(\lambda)$ is generated by v^+ we have that $Z(\mathfrak{g})$ acts on all of $M(\lambda)$ as multiplication by χ_λ.

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For $M \in \mathcal{O}$ and $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ define
$M^\chi := \{v \in M | \forall z \in Z. \exists n : (z - \chi(z))^n v = 0\}$ i.e. z acts locally as multiplication by $\chi(z)$ plus a nilpotent operator.

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It is easy to check that the M^χ are submodules of M and are all independent.

Decomposition of $\mathcal{O}$

Proposition

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Proposition

Category $\mathcal{O}$ is the direct sum of the subcategories $\mathcal{O}_\chi$. Therefore each indecomposable lies in a unique $\mathcal{O}_\chi$. In particular, each highest weight module of weight λ lies in $\mathcal{O}_{\chi_\lambda}$

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When is $\chi_\lambda = \chi_\mu$? Are all χ of the form χ_λ?

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Theorem (Harish-Chandra)

1. *$\forall \lambda, \mu \in \mathfrak{h}^*$ we have $\chi_\lambda = \chi_\mu$ iff $\exists w \in W : \mu = w \cdot \lambda$*
2. *Every central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is of the form χ_λ*

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Conclusion

The subcategories $\mathcal{O}_\chi$ each contain a finite number of simple modules $L(\lambda)$, i.e. those $L(\lambda)$ such that $\lambda \in W \cdot \lambda_0$, where $\chi = \chi_{\lambda_0}$.

The categories $\mathcal{O}_\chi$

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Remark

Each $M \in \mathcal{O}$ possesses a composition series $0 = M_0 \subset \ldots \subset M_n = M$ s.t $M_i/M_{i-1} \cong L(\lambda)$ and $[M : L(\lambda)]$ is well defined.
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Thus we want to study the structure of $L(\lambda) \in \mathcal{O}_\chi$. This together with the decomposition series will give us substantional information about the structure of $M \in \mathcal{O}$. This leads to the notion of **formal characters** in $\mathcal{O}$.

Characters in $\mathcal{O}$

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Remark:

$0 \to M' \to M \to M'' \to 0 \Rightarrow \mathrm{ch}M = \mathrm{ch}M' + \mathrm{ch}M''$

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Computing $\text{ch}_{L(\lambda)}$ directly is difficult. On the other hand $\text{ch}_{M(\lambda)}$ is given by a simple formula (since it is a free $U(\mathfrak{n}_-)$ module) and they turn out to be closely related.

Characters in $\mathcal{O}$

Observation

$\mathrm{ch}_{M(\lambda)} = \sum_\mu a(\lambda,\mu)\mathrm{ch}_{L(\mu)}$
Here $\mu \leq \lambda$ and linked to λ, $a(\lambda,\mu) = [M(\lambda) : L(\mu)] \in \mathbb{Z}^+$ and $a(\lambda,\lambda) = 1$.

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Inverting this triangular linear system, we get:
$\mathrm{ch}_{L(\lambda)} = \sum_\mu b(\lambda, \mu)\mathrm{ch}_{M(\mu)} \Leftrightarrow \mathrm{ch}_{L(\lambda)} = \sum_{w\cdot\lambda \leq \lambda} b(\lambda, w)\mathrm{ch}_{M(w\cdot\lambda)}$

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Remark

Using the above observation it is possible to use formal characters to prove the Weyl charcter formula.

Kazhdan-Lustig Conjecture

Consider the subcategory $\mathcal{O}_0 := \mathcal{O}_{\chi_0}$ (**the principal block**). The weight -2ρ is minimal in this linkage class. Note that $M(-2\rho) = L(-2\rho)$.
We parametrize the simple and Verma modules in $\mathcal{O}_0$ by the elements of W, e.g. write $L_w := L(w \cdot (-2\rho))$. It holds that $[M_w : L_x] \neq 0$ iff $x \leq w$ in the Bruhat ordering of the Weyl group.

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Conjecture (Kazhdan-Lustig)

$$\mathrm{ch}_{L_w} = \sum_{x \leq w} (-1)^{l(w)-l(x)} P_{x,w}(1) \mathrm{ch}_{M_x}$$

Where $P_{x,w}$ is a Kazhdan-Lustig polynomial.

Kazhdan-Lustig Conjecture

Remark

For regular integral weights λ(i.e. $\lambda \in \Lambda : |W \cdot \lambda| = |W|$) the categories $\mathcal{O}_{\chi_\lambda}$ are equivalent, so this result allows to describe them as well.
There are similar results for the rest of $\mathcal{O}$ involving certain subgroups of W.

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Theorem

For each simple module $L(\lambda) \in \mathcal{O}$ there is a unique indecomposable projective $P(\lambda) \in \mathcal{O}$ with an epimorphism $P(\lambda) \to L(\lambda)$. Moreover, we can decompose this epimorphism as two epimorphisms $P(\lambda) \to M(\lambda) \to L(\lambda)$.

Verma flags and BGG Reciprocity

For each category $\mathcal{O}_\chi$ this gives a matrix with entries $[P(\lambda) : L(\mu)]$ called the **Cartan matrix** of the category.
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Theorem (BGG Reciprocity)

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Remark: $\mathcal{O}_\lambda$ for $\lambda \notin \mathbb{Z}$ in this case decomposes as a direct sum of two categories each containing one irreducible module.