Parity Sheaves and Moment Graphs

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Outline

Setting

$T \cong (\mathbb{C}^*)^r$: a complex torus.

X: a complete normal complex T-variety.

Assumptions:

(A1a) number of zero- and one-dimensional orbits finite;

(A1b) the closure of each one-dimensional orbit is smooth (and hence isomorphic to $\mathbb{P}^1\mathbb{C}$).

Unoriented Moment Graph of a T-Variety

$\mathcal{G}_X = (\mathcal{V}, \mathcal{E}, \alpha)$: unoriented moment graph over the character lattice $X^*(T)$:

vertices: $\mathcal{V} = X^T =$ set of T-fixed points in X.

edges: $\mathcal{E} =$ set of one-dimensional orbits in X. An edge connects the two (different) fixed points in its closure.

labels: If E is an edge, let $\alpha_E \in X^*(T)$ be the character such that $\ker \alpha_E \subset T$ is the stabilizer of some/any point in E.
α_E is well defined up to sign.

Reminder: Equivariant Derived Category I

$\mathcal{D}^{\mathrm{b}}_T(X) = \mathcal{D}^{\mathrm{b}}_T(X; k)$: T-equivariant derived category of sheaves of modules over a commutative ring k.

Let $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_T(X)$.

- The equivariant cohomology of X with coefficients in $\mathcal{F}$ is

$$\mathbb{H}_T(\mathcal{F}) = H(BT; \pi_* \mathcal{F}),$$

  where $\pi_* : \mathcal{D}^{\mathrm{b}}_T(X; k) \to \mathcal{D}^{\mathrm{b}}_T(\mathrm{pt}; k) \subset \mathcal{D}^{\mathrm{b}}(BT; k)$ comes from $\pi : X \to \mathrm{pt}$.
- $\mathbb{H}_T(\mathcal{F})$ is a graded module over

$$S_k := S_k(X^*(T)_k) = H(BT; k_{BT}) = \mathbb{H}_T(k_{\mathrm{pt}}).$$

Reminder: Equivariant Derived Category II

- For $i : Y \subset X$ a T-equivariant embedding define

$$\mathcal{F}_Y := i^*\mathcal{F}.$$

The adjunction map $\mathcal{F} \to i_*\mathcal{F}_Y$ yields the restriction map for equivariant cohomology

$$\mathbb{H}_T(\mathcal{F}) \to \mathbb{H}_T(\mathcal{F}_Y).$$

Write $\mathcal{F}_x := \mathcal{F}_{\{x\}}$ if $Y = \{x\}$ is a fixed point.

Basic Example.

Let T act on $\mathbb{C}$ linearly via a non-trivial character α.

[a] We assume that $n|\alpha$ in $X^*(T)$ implies that n is invertible in k.

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- For $k_{\mathbb{C}}$ the constant equivariant sheaf on $\mathbb{C}$, the maps[a]

$$\mathbb{H}_T(k_{\mathbb{C}}) \to \mathbb{H}_T(k_0) \quad \text{resp.} \quad \mathbb{H}_T(k_{\mathbb{C}}) \to \mathbb{H}_T(k_{\mathbb{C}^*})$$

  are identified with

$$S_k \xrightarrow{\sim} S_k \quad \text{resp.} \quad S_k \twoheadrightarrow S_k/(\alpha)$$

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- Let $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_T(\mathbb{C})$ and consider $\{0\} \xrightarrow{i} \mathbb{C} \xrightarrow{\pi} \{0\}$. Applying π_* to the adjunction morphism $\mathcal{F} \to i_* i^* \mathcal{F}$ yields an isomorphism

$$\pi_* \mathcal{F} \xrightarrow{\sim} \pi_* i_* i^* \mathcal{F} = i^* \mathcal{F} \quad \text{in } \mathcal{D}^{\mathrm{b}}_T(\{0\}).$$

  In particular $\mathbb{H}_T(\mathcal{F}) \xrightarrow{\sim} \mathbb{H}_T(\mathcal{F}_0)$.

[a] We assume that $n|\alpha$ in $X^*(T)$ implies that n is invertible in k.

Definition of the Functor $\mathbb{W}$

$\mathcal{G}_X\text{-}\mathrm{gMod}_k$: category of graded sheaves on $\mathcal{G}_X$, coefficients in k (all stalks S_k-modules).

$$\mathbb{W} : \mathcal{D}^{\mathrm{b}}_T(X; k) \to \mathcal{G}_X\text{-}\mathrm{gMod}_k\,.$$

Let $\mathcal{F}$ in $\mathcal{D}^{\mathrm{b}}_T(X; k)$.

- $x \in \mathcal{V} = X^T$ a vertex:

$$\mathbb{W}(\mathcal{F})^x := \mathbb{H}_T(\mathcal{F}_x)$$

- $E \subset X$ an edge:

$$\mathbb{W}(\mathcal{F})^E := \mathbb{H}_T(\mathcal{F}_E).$$

These are graded S_k resp. $S_k/(\alpha_E)$-modules.

Definition of the Functor $\mathbb{W}$

- x a fixed point in the closure of an orbit E:
  Then $E \cup \{x\}$ is isomorphic to $\mathbb{C}$ with T acting by $\pm\alpha_E$.
  The restriction morphisms on equivariant cohomology and the Basic Example yield

$$\mathbb{H}_T(\mathcal{F}_{E\cup\{x\}}) \longrightarrow \mathbb{H}_T(\mathcal{F}_E) = \mathbb{W}(\mathcal{F})^E$$

$$\mathbb{H}_T(\mathcal{F}_{E\cup\{x\}}) \xrightarrow{\sim} \mathbb{H}_T(\mathcal{F}_x) = \mathbb{W}(\mathcal{F})^x$$

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$$
\begin{array}{ccc}
 & & \mathbb{H}_T(\mathcal{F}_E) = \mathbb{W}(\mathcal{F})^E \\
 & \nearrow & \\
\mathbb{H}_T(\mathcal{F}_{E\cup\{x\}}) & \circlearrowleft & \uparrow \rho_{x,E} \\
 & \searrow \sim & \\
 & & \mathbb{H}_T(\mathcal{F}_x) = \mathbb{W}(\mathcal{F})^x
\end{array}
$$

Localization Theorem I

We impose on our T-variety X in addition to (A1a+b) the following assumptions:

(A2) Each fixed point is attractive: If x is a fixed point, there is an open neighborhood U of x in X and a 1-parameter subgroup $\chi : \mathbb{C}^* \to T$ such that

$$\lim_{z \to 0} \chi(z).u = x \quad \text{for all } u \in U.$$

(A3a) For any $\alpha \in X^*(T)$, the unoriented moment graph $\mathcal{G}_X^\alpha$ that is obtained from $\mathcal{G}_X$ by deleting all edges E with $k\alpha \cap k\alpha_E = 0$, is a disjoint union of moment graphs with only one or two vertices.

(A3b) If E is a one-dimensional orbit and $n \in \mathbb{Z}$ is such that α_E is divisible by n in $X^*(T)$, then n is invertible in k.

Localization Theorem II

Theorem (Localization Theorem)

Let X be a complete normal T-variety satisfying (A1a)-(A3b) and $\mathcal{F}$ be in $\mathcal{D}^{\mathrm{b}}_T(X;k)$. If $\mathbb{H}_T(\mathcal{F})$ and $\mathbb{H}_T(\mathcal{F}_{X^T})$ are free S_k-modules, then the restriction morphism $\mathbb{H}_T(\mathcal{F}) \to \mathbb{H}_T(\mathcal{F}_{X^T})$ is injective, and

$$\mathbb{H}_T(\mathcal{F}) = \Gamma(\mathbb{W}(\mathcal{F}))$$

as submodules of $\mathbb{H}_T(\mathcal{F}_{X^T}) = \bigoplus_{x \in X^T} \mathbb{H}_T(\mathcal{F}_x) = \bigoplus_{x \in X^T} \mathbb{W}(\mathcal{F})^x$.

Oriented Moment Graph of a stratified T-Variety I

X a T-variety satisfying conditions (A1a+b) and (A2). We assume that X is endowed with a Whitney stratification

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda.$$

by T-stable locally closed subvarieties, such that the closure of each stratum is a union of strata. Assume that

(S) For each $\lambda \in \Lambda$ there is a T-equivariant isomorphism $X_\lambda \cong \mathbb{C}^{n_\lambda}$, where $\mathbb{C}^{n_\lambda}$ carries a linear T-action.

We obtain a bijection

$$\Lambda \xrightarrow{\sim} X^T = \{\text{vertices of } \mathcal{G}_X\},$$
$$\lambda \mapsto x_\lambda.$$

Partial order on Λ:

$$\lambda \leq \mu \text{ if and only if } X_\lambda \subset \overline{X}_\mu.$$

This turns $\mathcal{G}_X = (\mathcal{V}, \mathcal{E}, \alpha)$ into an ordered moment graph.
If $\lambda \leq \mu$ and an edge E connects $x_\lambda = \lambda$ and $x_\mu = \mu$, we direct E as follows:

$$E : x_\lambda \to x_\mu.$$

Alexandrov^{op} topology on $\mathcal{G}_X$: Basis of open subsets is $\{\geq \lambda\}_{\lambda \in \Lambda}$.

Braden-MacPherson Sheaves I

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha)$: finite directed moment graph over a lattice Y.

k: commutative local noetherian ring.

$S_k = S_k(Y \otimes_{\mathbb{Z}} k)$.

Braden-MacPherson Sheaves II

Definition

A sheaf $\mathscr{B}$ on a directed moment graph $\mathcal{G}$ is called a **Braden-MacPherson sheaf** or **BM-sheaf**, if it satisfies the following conditions:

1. $\mathscr{B}^x$ is a graded-free S_k-module, for any $x \in \mathcal{V}$.
2. For any directed edge $E : x \to y$, the map $\rho_{y,E} : \mathscr{B}^y \to \mathscr{B}^E$ is surjective with kernel $\alpha_E \mathscr{B}^y$.
3. The sheaf $\mathscr{B}$ is flabby in the Alexandrov$^{\mathrm{op}}$ topology: For any open subset $\mathcal{J}$ of $\mathcal{V}$, the map
$$\Gamma(\mathscr{B}) = \Gamma(\mathcal{V}; \mathscr{B}) \to \Gamma(\mathcal{J}; \mathscr{B})$$
is surjective.
4. The map $\Gamma(\mathscr{B}) \to \mathscr{B}^x$ is surjective for any $x \in \mathcal{V}$.

Theorem

1. *For any $w \in \mathcal{V}$ there is an up to isomorphism unique BM-sheaf $\mathscr{B}(w)$ on $\mathcal{G}$ with the following properties:*
   - $\mathscr{B}(w)$ *has support in* $\{\leq w\}$.
   - $\mathscr{B}(w)^w \cong S_k$.
   - $\mathscr{B}(w)$ *is indecomposable in* $\mathcal{G}_X\text{-}\mathrm{gMod}_k$.
2. *Any BM-sheaf of finite type decomposes into a finite direct sum of objects of the form $\mathscr{B}(w)[l]$, for suitable $w \in \mathcal{V}$ and $l \in \mathbb{Z}$. This decomposition is unique up to permutation and isomorphism.*

Parity Sheaves and Braden-MacPherson Sheaves I

k: complete local principal ideal domain.

X: complex T-variety, satisfying (A1a)-(A3b), (S).

We further assume that for each $\lambda \in \Lambda$, there is a T-equivariant surjective morphism (not assumed to be birational)

$$\pi_\lambda : \widetilde{X}_\lambda \to \overline{X}_\lambda$$

such that

(R1) $\widetilde{X}_\lambda$ is a smooth projective T-variety;

(R2) $(\widetilde{X}_\lambda)^T$ is finite;

(R3) the derived direct image $\pi_{\lambda *} k_{\widetilde{X}_\lambda}$ lies in $\mathcal{D}^{\mathrm{b}}_{T,\Lambda}(X; k)$.

Parity Sheaves and Braden-MacPherson Sheaves II

Theorem (Fiebig, Williamson)

The functor $\mathbb{W} : \mathcal{D}^{\mathrm{b}}_T(X; k) \to \mathcal{G}_X\text{-}\mathrm{gMod}_k$ *restricts to a fully faithful functor*

$$\mathbb{W} : \{\textit{parity sheaves}\} \to \{\textit{BM-sheaves}\}$$

mapping $\mathcal{P}(\lambda)$ *to* $\mathscr{B}(\lambda)$*.*

Proof.

Let $\mathcal{P} \in \mathcal{D}^{\mathrm{b}}_{T}(X;k)$ be a parity sheaf. We check that $\mathbb{W}(\mathcal{P})$ satisfies the four defining properties of a BM-sheaf.
1: By definition, $\mathcal{P}_{X_\lambda}$ is a direct sum of shifted equivariant constant sheaves, and the same holds for $\mathcal{P}_{x_\lambda}$. This implies that $\mathbb{W}(\mathcal{P})_{x_\lambda} = \mathbb{H}_T(\mathcal{P}_{x_\lambda})$ is graded-free.
This also shows that $\mathbb{H}_T(\mathcal{P}_{X^T})$ is a graded-free S_k-module.

Proof continued.

3: Let $\mathcal{J} \subset \mathcal{G}$ be an open subset and $X_{\mathcal{J}}$ the corresponding open union of strata. Let $i : X \setminus X_{\mathcal{J}} \hookrightarrow X$ be the inclusion of its complement.

The "Hom(*-even, !-even)"-short exact sequence for parity sheaves with k_X the equivariant constant sheaf as its left argument is

$$0 \to \mathbb{H}_T(i^!\mathcal{P}) \to \mathbb{H}_T(\mathcal{P}) \to \mathbb{H}_T(\mathcal{P}_{X_{\mathcal{J}}}) \to 0.$$

An induction on the stratification yields that $\mathbb{H}_T(\mathcal{P})$ is a free S_k-module. The Localization Theorem identifies the above epimorphism with the map

$$\Gamma(\mathbb{W}(\mathcal{P})) \to \Gamma(\mathcal{J}; \mathbb{W}(\mathcal{P})).$$

Proof continued.

2: Let $E : x_\mu \to x_\lambda$ be a directed edge. Then the one-dimensional orbit E is contained in X_λ. The map

$$\rho_{x_\lambda, E} : \mathbb{W}(\mathcal{P})^{x_\lambda} \to \mathbb{W}(\mathcal{P})^E$$

is defined by

$$\mathbb{H}_T(\mathcal{F}_{E \cup \{x\}}) \to \mathbb{H}_T(\mathcal{F}_E).$$

Since $\mathcal{P}_{X_\lambda}$ is a direct sum of shifted equivariant constant sheaves, we only have to prove that

$$\mathbb{H}_T(k_{E \cup \{x\}}) \to \mathbb{H}_T(k_E)$$

is surjective with kernel $\alpha_E \mathbb{H}_T(k_{E \cup \{x\}})$. We have seen this in the above Basic Example.

Proof continued.

4: Omitted. Uses (R1)-(R3).

The fully faithfulness follows from the “Hom(*-even, !-even)”-short exact sequence for parity sheaves and an induction on the number of strata. This implies that $\mathbb{W}(\mathcal{P}(\lambda))$ is indecomposable. Hence, $\mathbb{W}(\mathcal{P}(\lambda)) \cong \mathscr{B}(\lambda)$ by considering the support.

□

Kazhdan-Lusztig Conjecture I

$k = \mathbb{C}$.

$\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$: semisimple complex Lie algebra with Borel and Cartan subalgebra.

$\Phi^+ \subset \Phi \subset \mathfrak{h}^*$: roots of $\mathfrak{b}$ and $\mathfrak{g}$ with respect to $\mathfrak{h}$.

W: Weyl group.

$\Delta(\mu)$: Verma module with highest weight $\mu \in \mathfrak{h}^*$.

$L(\mu)$: the simple quotient of $\Delta(\mu)$.

$P(\mu)$: a projective cover of $L(\mu)$ in $\mathcal{O}$.

Theorem (Kazhdan-Lusztig-Conjecture)

Let $\lambda \in \mathfrak{h}^*$ be regular integral antidominant. Then

$$[\Delta(x \cdot \lambda) : L(y \cdot \lambda)] = h_{xw_0, yw_0}(1)$$

for all $x, y \in W$.

Proof.

$S = S(\mathfrak{h})$

$A = S_{S\mathfrak{h}}$: localization of S at the maximal ideal generated by $\mathfrak{h}$.

$\mathcal{G}^\lambda$: ordered moment graph over the root lattice:

- set of vertices: W
- edges: $w, w' \in W$, $\alpha \in \Phi^+$:

$$w \overset{\alpha}{\text{———}} w' \iff w = s_\alpha w'.$$

- ordering: $w \leq^\lambda w' \iff w \cdot \lambda \leq w' \cdot \lambda$. (Refines Bruhat ordering.)

$\mathcal{G}^{\text{Bruhat}}$: moment graph with the same vertices, edges and labels, but Bruhat ordering.

$\mathscr{B}^{\text{op}}(x)$: BM sheaves (with $\mathbb{C}$ coefficients) on $(\mathcal{G}^\lambda)^{\text{op}}$ and $(\mathcal{G}^{\text{Bruhat}})^{\text{op}}$.
$\mathscr{B}^{\text{op}}(x)$ has support in $\{\geq x\}$.

Proof continued.

$$\begin{aligned}[\Delta(x\cdot\lambda):L(y\cdot\lambda)] &= (P(y\cdot\lambda):\Delta(x\cdot\lambda)) \\ &= (P_A(y\cdot\lambda):\Delta_A(x\cdot\lambda)) \\ &= \operatorname{rank}_A(\mathcal{L}\mathbb{V}P_A(y\cdot\lambda))^x \\ &= \operatorname{rank}_A(\mathscr{B}^{\mathrm{op}}(y)\otimes_S A)^x \\ &= \operatorname{rank}_S \mathscr{B}^{\mathrm{op}}(y)^x\end{aligned}$$

Proof continued.

Geometry enters the game.

$G^\vee \supset B^\vee \supset T^\vee$: semisimple connected complex algebraic group with Borus such that the root system of $(G^\vee, T^\vee)$ is dual to $\Phi \subset \mathfrak{h}^*$ and such that the roots of $B^\vee$ are $\Phi^{+\vee} = (\Phi^\vee)^+$.

$X = G^\vee / B^\vee$: the flag variety, considered as a $T^\vee$-variety.

$X_w = B^\vee w B^\vee / B^\vee$: the Bruhat cell associated to $w \in W$.

$X = \bigsqcup_{w \in W} X_w$: stratification into Bruhat cells.

$\mathcal{G}_X$: associated oriented moment graph over $X^*(T^\vee)$. Coincides with $\mathcal{G}^{\text{Bruhat}}$.

All our assumptions satisfied.

Proof continued.

$\mathscr{B}(x)$: BM-sheaves (with $\mathbb{C}$-coefficients) on $\mathcal{G}_X$.
$\mathscr{B}(x)$ has support in $\{\leq x\}$.

Note that

$$X^*(T^\vee)_\mathbb{C} = (\text{Lie } T^\vee)^* = \mathfrak{h}$$

implies

$$S(X^*(T^\vee)_\mathbb{C} = S(\mathfrak{h}) = S.$$

Proof continued.

The isomorphism of directed moment graphs

$$(\mathcal{G}^{\text{Bruhat}})^{\text{op}} \xrightarrow{\sim} \mathcal{G}_X, \quad x \mapsto xw_0.$$

identifies $\mathscr{B}^{\text{op}}(y)$ and $\mathscr{B}(yw_0)$. Hence

$$\begin{aligned}[\Delta(x \cdot \lambda) : L(y \cdot \lambda)) &= \text{rank}_S\, \mathscr{B}^{\text{op}}(y)^x \\ &= \text{rank}_S\, \mathscr{B}(yw_0)^{xw_0}\end{aligned}$$

Proof continued.

By the above Theorem $\mathbb{W}(\mathcal{P}(yw_0)) \cong \mathscr{B}(yw_0)$.
Since we work with complex coefficients, we have $\mathcal{P}(yw_0) \cong \mathcal{IC}_T(\overline{X}_{yw_0})$. Hence

$$\begin{aligned}\mathscr{B}(yw_0)^{xw_0} &\cong \mathbb{H}_T(\mathcal{P}(yw_0)_{xw_0}) \\ &\cong \mathbb{H}_T(\mathcal{IC}_T(\overline{X}_{yw_0})_{xw_0}).\end{aligned}$$

All these modules being S-free, we have

$$\mathbb{C} \otimes_S \mathbb{H}_T(\mathcal{IC}_T(\overline{X}_{yw_0})_{xw_0}) \cong \mathbb{H}(\mathcal{IC}(\overline{X}_{yw_0})_{xw_0})$$

and finally

$$\begin{aligned}[\Delta(x \cdot \lambda) : L(y \cdot \lambda)] &= \operatorname{rank}_S \mathscr{B}(yw_0)^{xw_0} \\ &= \dim_{\mathbb{C}} \mathbb{H}(\mathcal{IC}(\overline{X}_{yw_0})_{xw_0}) \\ &= h_{xw_0,yw_0}(1).\end{aligned}$$

□

Many thanks to the organizers!