Parity Sheaves

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Sheaves in Representation Theory
Isle of Skye

Introduction

Motivation. Use geometric and homological methods in modular representation theory, namely sheaves with coefficients in a field k

- $\text{char}(k) = 0$: Kazhdan-Lusztig conjecture can be proved using IC complexes and the decomposition theorem
- $\text{char}(k) = p$: IC complexes exist but the decomposition theorem is no longer true

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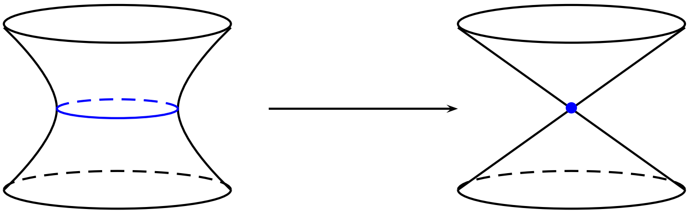
Observation. In the flag variety (as in many other situations)

$$\mathcal{H}^i(\mathrm{IC}(\overline{X}_w, k)) = 0 \text{ unless } i + \ell(w) \text{ is even}$$

when $\mathrm{char}(k) = 0$. But not always true in prime characteristic (torsion).

Failure of the decomposition theorem in $\mathfrak{sl}_2$

Springer resolution of $\mathcal{N} = \{(a,b,c) \in \mathbb{C}^3 \mid a^2 - bc = 0\}$



Stalks of $R\pi_*(\underline{k}[2])$

	-2	-1	0
$\mathcal{O}_{\text{reg}}$	k	0	0
$\{0\}$	k	0	k

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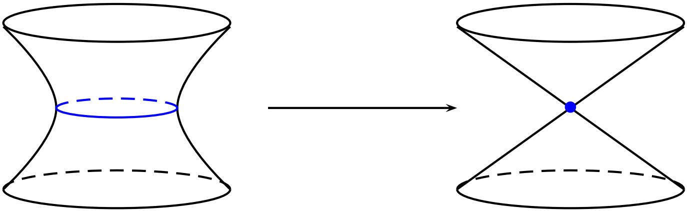

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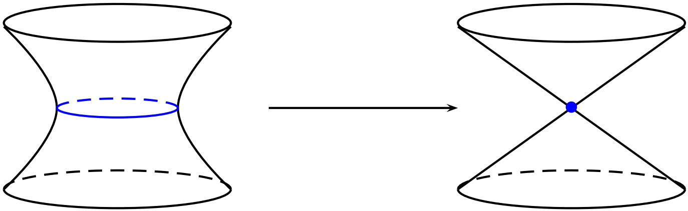

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Setting. X complex algebraic variety with Whitney stratification

$$X = \coprod_{\lambda \in \Lambda} X_\lambda$$

$D_c(X, k) =$ derived category of constructible k-sheaves

Definition

A even sheaf $\mathcal{F}$ is any bounded complex in $D_c(X, k)$ such that

$$\mathcal{H}^i(\mathcal{F}) = \mathcal{H}^i(\mathbb{D}\mathcal{F}) = 0 \quad \text{for odd } i$$

Equivalently, the stalks and costalks are concentrated in even degrees.

$\mathcal{F}$ is parity if it is a direct sum $\mathcal{F}' \oplus \mathcal{F}''[1]$ with $\mathcal{F}'$ and $\mathcal{F}''$ even.

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Remark. can adapt the definition when k is a local ring.

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Aim. Find a good substitute for IC complexes in prime characteristic

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- G-orbits in the nilpotent cone of $\mathfrak{g}$

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Theorem (Juteau-Mautner-Williamson)

Given an indecomposable local system $\mathcal{L}$ on X_λ there exists, up to isomorphism, at most one indecomposable parity complex $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ such that

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As a consequence $\mathbb{D}\mathcal{E}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathcal{E}(\overline{X}_\lambda, \mathcal{L}^\vee)$

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Even resolutions

A morphism $\pi : Y = \amalg Y_\mu \longrightarrow X = \amalg X_\lambda$ is a stratified morphism if

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This gives existence of parity sheaves for

- Flag varieties stratified by Schubert cells
- GL_n-orbits in the nilpotent cone of $\mathfrak{gl}_n$

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Decomposition theorem

For proving the existence, one needs an "analogue" of the decomposition theorem for parity sheaves:

Theorem-Observation (Juteau-Mautner-Williamson)

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Consequences. if Y is smooth

$$\pi_* \underline{k}_Y[\dim Y] \simeq \bigoplus \mathcal{E}(\overline{X}_{\lambda_i}, \mathcal{L}_i)[c_i]$$

In particular, if $\mathrm{char}(k) = 0$, the usual decomposition theorem shows that the IC complexes occurring are parity sheaves.

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- "Decomposition theorem in any characteristic"