# Computation of stalks of simple perverse sheaves on the flag variety (after MacPherson and Springer)

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# Table of contents

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## Hecke algebra

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## Definition (Hecke algebra of $(W, S)$)

*The Hecke algebra of $(W, S)$ is*

$$\mathcal{H}_W = \bigoplus_{w \in W} \mathbb{Z}[t, t^{-1}] \cdot T_w,$$

*with the multiplication given by*

$$\begin{cases} T_v \cdot T_w = T_{vw} & \text{if } \ell(vw) = \ell(v) + \ell(w), \\ T_s T_w = (t^2 - 1) T_w + t^2 T_{sw} & \text{if } s \in S \text{ and } sw < w. \end{cases}$$

# KL basis and KL polynomials

Kazhdan-Lusztig involution $i : \mathcal{H}_W \to \mathcal{H}_W$ is the algebra involution defined by the formulas

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**Theorem (Kazhdan-Lusztig)**

*For any $w \in W$, there exists a unique element $C_w \in \mathcal{H}_W$ which satisfies the following properties:*

1. $i(C_w) = C_w$
2. $C_w = t^{-\ell(w)} \sum_{x \leq w} Q_{x,w}(t) T_x$, *where* $Q_{w,w} = 1$ *and for* $x < w$, $Q_{x,w} \in \mathbb{Z}[t]$ *is a polynomial of degree* $\leq \ell(w) - \ell(x) - 1$.

*Moreover, for each $x \leq w$, there exists a polynomial $P_{x,w} \in \mathbb{Z}[q]$ (of degree $\leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$) such that $Q_{x,w}(t) = P_{x,w}(t^2)$.*

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- For $s \in S$ we have $C_s = t^{-1}(1 + T_s)$. Hence $P_{1,s} = 1$.
- For $s \neq t$ in $S$ we have $C_{st} = C_s C_t = t^{-2}(1 + T_s + T_t + T_{st})$.

# Bruhat decomposition and Schubert varieties

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$G$-equivariant versions:

$$\mathcal{B} \times \mathcal{B} = \bigsqcup_{w \in W} \mathfrak{Y}_w \qquad \text{with} \quad \mathfrak{Y}_w := G \cdot (B/B, wB/B),$$

$$\mathfrak{X}_w := \overline{\mathfrak{Y}_w} = \bigsqcup_{v \leq w} \mathfrak{Y}_v.$$

For any $y, w$ in $W$ and $i \in \mathbb{Z}$ we have

$$H^i(\mathrm{IC}(\mathfrak{X}_w)_{(B/B, yB/B)}) \;\cong\; H^{i+\dim(\mathcal{B})}(\mathrm{IC}(X_w)_{yB/B}).$$

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- *If* $s \in S$ *we have* $X_s = P_s/B \cong \mathbb{P}^1$, *hence* $\mathrm{IC}(X_s) = \underline{\mathbb{Q}}_{X_s}[1]$. *Similarly,* $\mathfrak{X}_s = \mathcal{B} \times_{\mathcal{P}_s} \mathcal{B}$.

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- *If* $s \in S$ *we have* $X_s = P_s/B \cong \mathbb{P}^1$*, hence* $\mathrm{IC}(X_s) = \underline{\mathbb{Q}}_{X_s}[1]$*. Similarly,* $\mathfrak{X}_s = \mathcal{B} \times_{\mathcal{P}_s} \mathcal{B}$*.*
- *For* $s \neq t$ *in* $S$*,* $X_{st}$ *is smooth, hence* $\mathrm{IC}(X_{st}) = \underline{\mathbb{Q}}_{X_{st}}[2]$*.*

# Bott-Samelson(-Demazure-Hansen) resolutions

Let $w \in W$, and let $w = s_1 \cdots s_n$ be a reduced decomposition.

$$BS_{(s_1,\cdots,s_n)} := P_{s_1} \times^B P_{s_2} \times^B \cdots \times^B P_{s_n}/B.$$

$$\varpi_{(s_1,\cdots,s_n)} : \left\{ \begin{array}{ccc} BS_{(s_1,\cdots,s_n)} & \to & \mathcal{B} \\ [p_1 : \cdots : p_n B/B] & \mapsto & p_1 \cdots p_n B/B \end{array} \right.$$

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$G$-equivariant versions:

$$\mathfrak{B}\mathfrak{S}_{(s_1,\cdots,s_n)} := \mathcal{B} \times_{\mathcal{P}_{s_1}} \cdots \times_{\mathcal{P}_{s_n}} \mathcal{B} \cong \mathfrak{X}_{s_1} \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathfrak{X}_{s_n}.$$

$$\pi_{(s_1,\cdots,s_n)} : \left\{ \begin{array}{ccc} \mathfrak{B}\mathfrak{S}_{(s_1,\cdots,s_n)} & \to & \mathcal{B} \times \mathcal{B} \\ (g_0 B/B, \cdots, g_n B/B) & \mapsto & (g_0 B/B, g_n B/B) \end{array} \right.$$

# Perverse sheaves on $\mathcal{B} \times \mathcal{B}$

$D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B})$: bounded derived category of sheaves of $\mathbb{Q}$-vector spaces on $\mathcal{B} \times \mathcal{B}$, constructible with respect to the stratification $\mathcal{S}$ by $G$-orbits.

Notation: $\mathcal{A}_w := \mathcal{A}_{(B/B, wB/B)}$.

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For $\mathcal{A}$ in $D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B})$, consider $h(\mathcal{A}) \in \mathcal{H}_W$ defined by the formula:

$$h(\mathcal{A}) \; = \; \sum_{w \in W} \Big( \sum_{i \in \mathbb{Z}} \dim H^i(\mathcal{A}_w) t^i \Big) \cdot T_w.$$

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Convolution product: for $\mathcal{A}_1, \mathcal{A}_2$ in $D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B})$,

$$\mathcal{A}_1 \star \mathcal{A}_2 \ := \ (p_{1,3})_* \big( p_{1,2}^* \mathcal{A}_1 \otimes_{\mathbb{Q}} p_{2,3}^* \mathcal{A}_2 \big) \quad \in D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B}).$$

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This product is associative, the unit is $\underline{\mathbb{Q}}_{\mathfrak{X}_1}$.

## Lemma (MacPherson, Springer)

*Let $\mathcal{A}$ be in $D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B})$, with $\mathcal{H}^i(\mathcal{A}) = 0$ if $i$ is odd (resp. even), and let $s \in S$. Then $\underline{\mathbb{Q}}_{\mathfrak{X}_s} \star \mathcal{A}$ has the same property, and*

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So, we have to prove that

$$\dim H^i((\underline{\mathbb{Q}}_{\mathfrak{X}_s} \star \mathcal{A})_w) = \begin{cases} \dim H^i(\mathcal{A}_{sw}) + \dim H^{i-2}(\mathcal{A}_w) & \text{if } sw < w, \\ \dim H^i(\mathcal{A}_w) + \dim H^{i-2}(\mathcal{A}_{sw}) & \text{if } sw > w. \end{cases}$$

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Assume $sw < w$. Let $\mathcal{C}$ be the restriction of $p_{1,2}^* \underline{\mathbb{Q}}_{\mathfrak{X}_s} \otimes_{\mathbb{Q}} p_{2,3}^* \mathcal{A}$ to

$$Z_w^s := \{(B/B, gB/B, wB/B), \ g \in P_s\} \cong \mathbb{P}^1.$$

We have $H^n((\underline{\mathbb{Q}}_{\mathfrak{X}_s} \star \mathcal{A})_w) \cong H^n(Z_w^s, \mathcal{C})$.

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We have $(gB/B, wB/B) \in \mathfrak{Y}_{sw}$ iff $gB = sB$. Let

$$i : \{(B/B, sB/B, wB/B)\} \hookrightarrow Z_w^s$$

be the inclusion, and let $j$ be the inclusion of the complement.

Consider the exact triangle $j_! j^* \mathcal{C} \to \mathcal{C} \to i_* i^* \mathcal{C} \xrightarrow{+1}$, and the associated long exact sequence

$$\cdots \to H^n_c(j_! j^* \mathcal{C}) \to H^n_c(\mathcal{C}) \to H^n_c(i_* i^* \mathcal{C}) \to H^{n+1}_c(j_! j^* \mathcal{C}) \to \cdots$$

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We have

$$H^n_c(i_* i^* \mathcal{C}) \cong H^n(\mathcal{A}_{sw}), \quad H^n_c(j_! j^* \mathcal{C}) \cong H^{n-2}(\mathcal{A}_w).$$

(Because $j^* \mathcal{C}$ has constant cohomology, with fiber $\mathcal{A}_w$, and $H^*_c(\mathbb{C}, \underline{\mathbb{Q}}) = \mathbb{Q}[-2]$.)

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(Because $j^* \mathcal{C}$ has constant cohomology, with fiber $\mathcal{A}_w$, and $H^*_c(\mathbb{C}, \underline{\mathbb{Q}}) = \mathbb{Q}[-2]$.)
Hence the long exact sequence splits into short exact sequences, and

$$\dim H^n\big((\underline{\mathbb{Q}}_{\mathfrak{X}_s} \star \mathcal{A})_w\big) = \dim H^n(\mathcal{A}_{sw}) + \dim H^{n-2}(\mathcal{A}_w).$$

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The case $sw > w$ is similar. $\square$

# Computation of fibers

**Theorem**

*For $w \in W$, we have*

$$h(\mathrm{IC}(\mathfrak{X}_w)) = t^{-\dim \mathcal{B}} \cdot C_w.$$

*Hence for $y \leq w$ and $i \in \mathbb{Z}$,* $\dim(H^i(\mathrm{IC}(X_w)_y))$ *is zero if $i + \ell(w)$ is odd, and is the coefficient of $q^{(i+\ell(w))/2}$ in $P_{y,w}(q)$ if $i + \ell(w)$ is even.*

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*Proof*: By induction on $\ell(w)$.

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*Proof*: By induction on $\ell(w)$.
Let $w = s_1 \cdots s_n$ be a reduced decomposition. Consider the resolution

$$\pi_{(s_1,\cdots,s_n)} : \mathfrak{BS}_{(s_1,\cdots,s_n)} \to \mathfrak{X}_w.$$

Now we have

$$(\pi_{(s_1,\cdots,s_n)})_* \underline{\mathbb{Q}}_{\mathfrak{BS}_{(s_1,\cdots,s_n)}} \cong \underline{\mathbb{Q}}_{\mathfrak{X}_{s_1}} \star \cdots \star \underline{\mathbb{Q}}_{\mathfrak{X}_{s_n}}.$$

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Indeed,

$$\mathfrak{BS}_{(s_1,\cdots,s_n)} \cong \mathfrak{BS}_{(s_1,\cdots,s_{n-1})} \times_{\mathcal{B}} \mathfrak{X}_{s_n}.$$

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The claim follows by induction.

Then, by the crucial lemma,

$$\begin{aligned} h((\pi_{(s_1,\cdots,s_n)})_* \underline{\mathbb{Q}}_{\mathfrak{BS}_{(s_1,\cdots,s_n)}}[n]) &= t^{-n}(1+T_{s_1})\cdots(1+T_{s_n}) \\ &= C_{s_1}\cdots C_{s_n}. \end{aligned}$$

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By the Decomposition Theorem,

$$(\pi_{(s_1,\cdots,s_n)})_* \underline{\mathbb{Q}}_{\mathfrak{BS}_{(s_1,\cdots,s_n)}}[n+\dim\mathcal{B}] \cong \bigoplus_{y\leq w} \mathrm{IC}(\mathfrak{X}_y)\otimes_{\mathbb{Q}} V_y,$$

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where the $V_y$'s are graded finite dimensional $\mathbb{Q}$-vector space, with $V_w=\mathbb{Q}$.
This object is stable under $\mathbb{D}$, hence $\dim(V_y^n)=\dim(V_y^{-n})$.

It follows that

$$h((\pi_{(s_1,\cdots,s_n)})_* \underline{\mathbb{Q}}_{\mathfrak{BS}_{(s_1,\cdots,s_n)}}[n+\dim \mathcal{B}]) = h(\mathrm{IC}(\mathfrak{X}_w)) + \sum_{y<w} Q_y(t) h(\mathrm{IC}(\mathfrak{X}_y)),$$

where $Q_y$ is a Laurent polynomial such that $Q_y(t) = Q_y(t^{-1})$.

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Hence $t^{\dim\mathcal{B}} h(\mathrm{IC}(\mathfrak{X}_w))$ is stable under $i$.

For $y < w$, we have

$$H^{i-\dim \mathcal{B}}(\mathrm{IC}(\mathfrak{X}_w)_y) = 0 \quad \text{if } i \notin [\![-\ell(w), -\ell(y) - 1]\!]$$

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**Corollary**

*The coefficients of $P_{y,w}$ are non-negative.*

# Example: type $\mathbf{B}_2$

Consider $w = s_1 s_2 s_1$, and the resolution

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We have

$$H^*(\mathbb{P}^1) \ = \ \mathbb{Q} \oplus \mathbb{Q}[-2].$$

The fibers of $\pi_* \mathrm{IC}(\mathfrak{BS}) = \pi_* \underline{\mathbb{Q}}_{\mathfrak{BS}}[7]$ are given by:

| $\dim(\mathfrak{X}_v)$ | $v$ | $-7$ | $-6$ | $-5$ |
|---|---|---|---|---|
| 7 | $s_1 s_2 s_1$ | $\mathbb{Q}$ | 0 | 0 |
| 6 | $s_2 s_1$ | $\mathbb{Q}$ | 0 | 0 |
| 6 | $s_1 s_2$ | $\mathbb{Q}$ | 0 | 0 |
| 5 | $s_2$ | $\mathbb{Q}$ | 0 | 0 |
| 5 | $s_1$ | $\mathbb{Q}$ | 0 | $\mathbb{Q}$ |
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Hence

$$\pi_* \mathrm{IC}(\mathfrak{BS}) \;\cong\; \mathrm{IC}(\mathfrak{X}_{s_1 s_2 s_1}) \oplus \mathrm{IC}(\mathfrak{X}_{s_1}).$$

Moreover,

$$\mathrm{IC}(\mathfrak{X}_{s_1 s_2 s_1}) = \underline{\mathbb{Q}}_{\mathfrak{X}_{s_1 s_2 s_1}}[7].$$

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$$C_{s_1} C_{s_2} C_{s_1} = t^{-3}(1 + T_{s_1})(1 + T_{s_2})(1 + T_{s_1})$$

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# Example: type $\mathbf{A}_3$

Consider $w = s_1 s_3 s_2 s_3 s_1$, and the resolution

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$\pi$ is an isomorphism over $\mathfrak{X}_w - \mathfrak{X}_{s_1 s_3}$, and all the non-trivial fibers are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. For example, we have

$$\pi^{-1}(B/B, B/B) = \{(B/B, gB/B, ghB/B, ghB/B, gB/B, B/B), \; g \in P_{s_3}, \; h \in P_{s_1}\}.$$

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We have

$$H^*(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Q} \oplus \mathbb{Q}^2[-2] \oplus \mathbb{Q}[-4].$$

The fibers of $\pi_* \mathrm{IC}(\mathfrak{B}\mathfrak{S}) = \pi_* \underline{\mathbb{Q}}_{\mathfrak{B}\mathfrak{S}}[11]$ are given by:

| $\dim(\mathfrak{X}_v)$ | $v$ | $-11$ | $-10$ | $-9$ | $-8$ | $-7$ |
|---|---|---|---|---|---|---|
| | $\mathcal{B}^2 - \mathfrak{X}_w$ | 0 | 0 | 0 | 0 | 0 |
| 11–7 | $\mathfrak{X}_w - \mathfrak{X}_{s_1 s_3}$ | $\mathbb{Q}$ | 0 | 0 | 0 | 0 |
| 8 | $s_1 s_3$ | $\mathbb{Q}$ | 0 | $\mathbb{Q}^2$ | 0 | $\mathbb{Q}$ |
| 7 | $s_1$ | $\mathbb{Q}$ | 0 | $\mathbb{Q}^2$ | 0 | $\mathbb{Q}$ |
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Hence

$$\pi_*\mathrm{IC}(\mathfrak{B}\mathfrak{S}) \;=\; \mathrm{IC}(\mathfrak{X}_w) \oplus \mathrm{IC}(\mathfrak{X}_{s_1 s_3})[1] \oplus \mathrm{IC}(\mathfrak{X}_{s_1 s_3})[-1].$$

Moreover, the fibers of $\mathrm{IC}(\mathfrak{X}_w)$ are given by:

| $\dim(\mathfrak{X}_v)$ | $v$ | $-11$ | $-10$ | $-9$ |
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| | $\mathcal{B}^2 - \mathfrak{X}_w$ | 0 | 0 | 0 |
| 11–7 | $\mathfrak{X}_w - \mathfrak{X}_{s_1 s_3}$ | $\mathbb{Q}$ | 0 | 0 |
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In particular, $\mathfrak{X}_w$ is not rationally smooth, and $\pi$ is not semi-small.

# Geometric realization of $\mathcal{H}_W$

Consider the subcategory $\mathcal{D}$ of $D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B})$ whose objects are the semisimple complexes, i.e. the complexes of the form

$$\bigoplus_{x \in W} \mathrm{IC}(\mathfrak{X}_x) \otimes_{\mathbb{Q}} V_x.$$

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We have

$$h : \mathrm{Obj}(\mathcal{D}) \to \mathcal{H}_W.$$

One can check that $\mathcal{D}$ is stable under the convolution, and that

$$h(\mathcal{A}_1 \star \mathcal{A}_2) = h(\mathcal{A}_1) \cdot h(\mathcal{A}_2)$$

for any $\mathcal{A}_1$, $\mathcal{A}_2$ in $\mathcal{D}$.

$\mathcal{D}$ is stable under shifts, and

$$h(\mathcal{A}[1]) = t^{-1} h(\mathcal{A})$$

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Finally, the image of $h$ is

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Hence $\mathcal{D}$ gives a geometric realization of $\mathcal{H}_W$.

*Remark*: It follows that for $x, y \in W$,

$$C_x \cdot C_y \in \bigoplus_{w \in W} \mathbb{Z}_{\geq 0}[t, t^{-1}] \cdot C_w.$$