Perverse Sheaves and the Decomposition Theorem

Delphine Dupont, Dragoş Frăţilă

May 26, 2010

Conventions

Throughout a variety will mean a quasi projective algebraic variety defined over the complex field. A topological space will be a paracompact, Hausdorff space.

Review on stratifications and pseudomanifolds

Definition

A *stratification* on a space X is a finite collection $\mathfrak{X}$ of locally closed subspaces of X called *strata* such that $X = \coprod_{S \in \mathfrak{X}} S$ and the closure of each stratum is a union of strata.

Definition

A *filtered space* X is a space together with a filtration by closed subsets $X = X_n \supseteq X_{n-1} \supseteq \ldots \supseteq X_0 \supseteq X_{-1} = \emptyset$.

Review on stratifications and pseudomanifolds

Definition

A *stratification* on a space X is a finite collection $\mathfrak{X}$ of locally closed subspaces of X called *strata* such that $X = \coprod_{S \in \mathfrak{X}} S$ and the closure of each stratum is a union of strata.

Definition

A *filtered space* X is a space together with a filtration by closed subsets $X = X_n \supseteq X_{n-1} \supseteq \dots \supseteq X_0 \supseteq X_{-1} = \emptyset$.

Review on stratifications and pseudomanifolds

Definition

Definition of a Whitney stratification.

Definition

If $\mathfrak{X}, \mathfrak{Y}$ are stratifications of X we say that $\mathfrak{Y}$ is a refinement of $\mathfrak{X}$ and we write $\mathfrak{X} \leq \mathfrak{Y}$ if every stratum $S \in \mathfrak{X}$ is a union of strata from $\mathfrak{Y}$.

Proposition

Let X be an algebraic variety. For any algebraic stratification $\mathfrak{X}$ of X there exists a refinement $\mathfrak{Y}$ which is an algebraic Whitney stratification.

Review on stratifications and pseudomanifolds

Definition

Definition of a Whitney stratification.

Definition

If $\mathfrak{X}, \mathfrak{Y}$ are stratifications of X we say that $\mathfrak{Y}$ is a refinement of $\mathfrak{X}$ and we write $\mathfrak{X} \leq \mathfrak{Y}$ if every stratum $S \in \mathfrak{X}$ is a union of strata from $\mathfrak{Y}$.

Proposition

Let X be an algebraic variety. For any algebraic stratification $\mathfrak{X}$ of X there exists a refinement $\mathfrak{Y}$ which is an algebraic Whitney stratification.

Review on stratifications and pseudomanifolds

Definition

Definition of a Whitney stratification.

Definition

If $\mathfrak{X}, \mathfrak{Y}$ are stratifications of X we say that $\mathfrak{Y}$ is a refinement of $\mathfrak{X}$ and we write $\mathfrak{X} \leq \mathfrak{Y}$ if every stratum $S \in \mathfrak{X}$ is a union of strata from $\mathfrak{Y}$.

Proposition

Let X be an algebraic variety. For any algebraic stratification $\mathfrak{X}$ of X there exists a refinement $\mathfrak{Y}$ which is an algebraic Whitney stratification.

Review on stratifications and pseudomanifolds

If not otherwise specified X will denote a complex algebraic variety with an algebraic Whitney stratification $\mathfrak{X}$. Throughout a sheaf will mean a sheaf of $\mathbb{Q}$-vector spaces.

Definition

A sheaf $\mathcal{F}$ on X is called constructible if $\mathcal{F}|_S$ is locally constant with stalks of finite dimension over $\mathbb{Q}$ for every $S \in \mathfrak{X}$.

Definition

The bounded derived category of constructible sheaves on X relative to $\mathfrak{X}$ is defined to be the full subcategory of the bounded derived category of sheaves on X^{an} such that their cohomology (as complexes) is constructible. We will denote this category by $\mathcal{D}^b_{\mathfrak{X}-c}(X)$.

Review on stratifications and pseudomanifolds

If not otherwise specified X will denote a complex algebraic variety with an algebraic Whitney stratification $\mathfrak{X}$. Throughout a sheaf will mean a sheaf of $\mathbb{Q}$-vector spaces.

Definition

A sheaf $\mathcal{F}$ on X is called constructible if $\mathcal{F}|_S$ is locally constant with stalks of finite dimension over $\mathbb{Q}$ for every $S \in \mathfrak{X}$.

Definition

The bounded derived category of constructible sheaves on X relative to $\mathfrak{X}$ is defined to be the full subcategory of the bounded derived category of sheaves on X^{an} such that their cohomology (as complexes) is constructible. We will denote this category by $\mathcal{D}^b_{\mathfrak{X}-c}(X)$.

Review on stratifications and pseudomanifolds

If not otherwise specified X will denote a complex algebraic variety with an algebraic Whitney stratification $\mathfrak{X}$. Throughout a sheaf will mean a sheaf of $\mathbb{Q}$-vector spaces.

Definition

A sheaf $\mathcal{F}$ on X is called constructible if $\mathcal{F}|_S$ is locally constant with stalks of finite dimension over $\mathbb{Q}$ for every $S \in \mathfrak{X}$.

Definition

The bounded derived category of constructible sheaves on X relative to $\mathfrak{X}$ is defined to be the full subcategory of the bounded derived category of sheaves on X^{an} such that their cohomology (as complexes) is constructible. We will denote this category by $\mathcal{D}^b_{\mathfrak{X}-c}(X)$.

Intermediate extension relative to a filtration

Let X be an algebraic variety with a filtration $\mathfrak{F} : X = X_n \supseteq ... \supseteq X_0 \supseteq X_{-1} = \emptyset$. We denote by

- $S_k = X_{n-k} \setminus X_{n-k-1}$
- $U_k = X \setminus X_{n-k-1}$, $k = 0, ..., n$.

We have inclusions

- $i_k : U_k \hookrightarrow U_{k+1}$ and
- $j_{k+1} : S_{k+1} \hookrightarrow U_{k+1}$

Remark that U_k is open in U_{k+1} and S_{k+1} is closed in U_{k+1}.

Intermediate extension relative to a filtration

Let X be an algebraic variety with a filtration $\mathfrak{F} : X = X_n \supseteq ... \supseteq X_0 \supseteq X_{-1} = \emptyset$. We denote by

- $S_k = X_{n-k} \setminus X_{n-k-1}$
- $U_k = X \setminus X_{n-k-1},\ k = 0, ..., n.$

We have inclusions

- $i_k : U_k \hookrightarrow U_{k+1}$ and
- $j_{k+1} : S_{k+1} \hookrightarrow U_{k+1}$

Remark that U_k is open in U_{k+1} and S_{k+1} is closed in U_{k+1}.

Intermediate extension relative to a filtration

Let X be an algebraic variety with a filtration $\mathfrak{F} : X = X_n \supseteq ... \supseteq X_0 \supseteq X_{-1} = \emptyset$. We denote by

- $S_k = X_{n-k} \setminus X_{n-k-1}$
- $U_k = X \setminus X_{n-k-1}$, $k = 0, ..., n$.

We have inclusions

- $i_k : U_k \hookrightarrow U_{k+1}$ and
- $j_{k+1} : S_{k+1} \hookrightarrow U_{k+1}$

Remark that U_k is open in U_{k+1} and S_{k+1} is closed in U_{k+1}.

Intermediate extension relative to a filtration

Let X be an algebraic variety with a filtration $\mathfrak{F} : X = X_n \supseteq ... \supseteq X_0 \supseteq X_{-1} = \emptyset$. We denote by

- $S_k = X_{n-k} \setminus X_{n-k-1}$
- $U_k = X \setminus X_{n-k-1},\ k = 0, ..., n.$

We have inclusions

- $i_k : U_k \hookrightarrow U_{k+1}$ and
- $j_{k+1} : S_{k+1} \hookrightarrow U_{k+1}$

Remark that U_k is open in U_{k+1} and S_{k+1} is closed in U_{k+1}.

Intermediate extension relative to a filtration

Definition

If X is an algebraic variety with a filtration $\mathfrak{F}$ as above, we define the intermediate extension relative to $\mathfrak{F}$ to be the functor

$$\iota_{!*}^{\mathfrak{F}} : Sh(S_0) \to \mathcal{D}^b(X)$$

defined by

$$\iota_{!*}^{\mathfrak{F}} := \tau_{\leq n-1} Ri_{n-1*} \circ \ldots \circ \tau_{\leq 0} Ri_{0*}$$

Intermediate extension relative to a filtration

$$\iota^{\mathfrak{F}}_{!*} := \tau_{\leq n-1} Ri_{n-1*} \circ \ldots \circ \tau_{\leq 0} Ri_{0*}$$

We have the following theorem of Deligne:

Theorem

The functor $\iota^{\mathfrak{F}}_{!*}$ establishes an equivalence of categories between the category $Sh(S_0)$ of sheaves on S_0 and the full subcategory $\mathfrak{IC}'_{\mathfrak{F}}(X) \subseteq \mathcal{D}^b(X)$ of complexes of sheaves $\mathcal{F}^\bullet$ that satisfy the following conditions

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < 0$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > 0$ *and for any* $k = 1, ..., n$ *the support and cosupport conditions*
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_k} = 0, \forall m \geq k$
- *(S')* $\mathcal{H}^m_{S_k}(\mathcal{F}^\bullet) = 0, \forall m \leq k$

Intermediate extension relative to a filtration

$$\iota_{!*}^{\mathfrak{F}} := \tau_{\leq n-1} Ri_{n-1*} \circ \ldots \circ \tau_{\leq 0} Ri_{0*}$$

We have the following theorem of Deligne:

Theorem

The functor $\iota_{!*}^{\mathfrak{F}}$ establishes an equivalence of categories between the category $Sh(S_0)$ of sheaves on S_0 and the full subcategory $\mathfrak{IC}'_{\mathfrak{F}}(X) \subseteq \mathcal{D}^b(X)$ of complexes of sheaves $\mathcal{F}^\bullet$ that satisfy the following conditions

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < 0$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > 0$ *and for any* $k = 1, \ldots, n$ *the support and cosupport conditions*
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_k} = 0, \forall m \geq k$
- *(S')* $\mathcal{H}^m_{S_k}(\mathcal{F}^\bullet) = 0, \forall m \leq k$

Intermediate extension relative to a filtration

$$\iota^{\mathfrak{F}}_{!*} := \tau_{\leq n-1} Ri_{n-1*} \circ \ldots \circ \tau_{\leq 0} Ri_{0*}$$

We have the following theorem of Deligne:

Theorem

The functor $\iota^{\mathfrak{F}}_{!*}$ establishes an equivalence of categories between the category $Sh(S_0)$ of sheaves on S_0 and the full subcategory $\mathfrak{IC}'_{\mathfrak{F}}(X) \subseteq \mathcal{D}^b(X)$ of complexes of sheaves $\mathcal{F}^\bullet$ that satisfy the following conditions

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < 0$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > 0$ *and for any* $k = 1, ..., n$ *the support and cosupport conditions*
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_k} = 0, \forall m \geq k$
- *(S')* $\mathcal{H}^m_{S_k}(\mathcal{F}^\bullet) = 0, \forall m \leq k$

Intermediate extension relative to a filtration

$$\iota_{!*}^{\mathfrak{F}} := \tau_{\leq n-1} Ri_{n-1*} \circ \ldots \circ \tau_{\leq 0} Ri_{0*}$$

We have the following theorem of Deligne:

Theorem

The functor $\iota_{!*}^{\mathfrak{F}}$ establishes an equivalence of categories between the category $Sh(S_0)$ of sheaves on S_0 and the full subcategory $\mathfrak{IC}'_{\mathfrak{F}}(X) \subseteq \mathcal{D}^b(X)$ of complexes of sheaves $\mathcal{F}^\bullet$ that satisfy the following conditions

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < 0$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > 0$ *and for any $k = 1, \ldots, n$ the support and cosupport conditions*
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_k} = 0, \forall m \geq k$
- *(S')* $\mathcal{H}^m_{S_k}(\mathcal{F}^\bullet) = 0, \forall m \leq k$

Intermediate extension relative to a filtration

$$\iota^{\mathfrak{F}}_{!*} := \tau_{\leq n-1} Ri_{n-1*} \circ ... \circ \tau_{\leq 0} Ri_{0*}$$

We have the following theorem of Deligne:

Theorem

The functor $\iota^{\mathfrak{F}}_{!*}$ establishes an equivalence of categories between the category $Sh(S_0)$ of sheaves on S_0 and the full subcategory $\mathfrak{IC}'_{\mathfrak{F}}(X) \subseteq \mathcal{D}^b(X)$ of complexes of sheaves $\mathcal{F}^\bullet$ that satisfy the following conditions

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < 0$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > 0$ *and for any* $k = 1, ..., n$ *the support and cosupport conditions*
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_k} = 0, \forall m \geq k$
- *(S')* $\mathcal{H}^m_{S_k}(\mathcal{F}^\bullet) = 0, \forall m \leq k$

Intermediate extension relative to a filtration

$$\iota_{!*}^{\mathfrak{F}} := \tau_{\leq n-1} R i_{n-1*} \circ \ldots \circ \tau_{\leq 0} R i_{0*}$$

We have the following theorem of Deligne:

Theorem

The functor $\iota_{!*}^{\mathfrak{F}}$ establishes an equivalence of categories between the category $Sh(S_0)$ of sheaves on S_0 and the full subcategory $\mathfrak{IC}'_{\mathfrak{F}}(X) \subseteq \mathcal{D}^b(X)$ of complexes of sheaves $\mathcal{F}^\bullet$ that satisfy the following conditions

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < 0$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > 0$ *and for any* $k = 1, ..., n$ *the support and cosupport conditions*
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_k} = 0, \forall m \geq k$
- *(S')* $\mathcal{H}^m_{S_k}(\mathcal{F}^\bullet) = 0, \forall m \leq k$

Intersection homology complexes

From now on we will denote by X a complex quasi-projective variety and by $\mathfrak{X}$ and $\mathfrak{F}$ an algebraic Whitney stratification and the associated filtration respectively. For a stratum S we will denote by d_S the complex dimension of the stratum.

Theorem

Using the above notations the intermediate extension functor $\iota^{\mathfrak{F}}_{!*}[d_X]$ establishes an equivalence of categories between $Loc(S_0)$ and the full subcategory of complexes $\mathcal{F}^\bullet$ in $\mathcal{D}^b_{\mathfrak{X}-c}(X)$ verifying the following conditions:

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < -d_X$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > -d_X$ *and* $\mathcal{H}^{-d_X}(\mathcal{F}^\bullet)|_{S_0} \in Loc(S_0)$
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$
- *(S')* $\mathcal{H}^m(\mathbb{D}_X\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$.

Intersection homology complexes

From now on we will denote by X a complex quasi-projective variety and by $\mathfrak{X}$ and $\mathfrak{F}$ an algebraic Whitney stratification and the associated filtration respectively. For a stratum S we will denote by d_S the complex dimension of the stratum.

Theorem

Using the above notations the intermediate extension functor $\iota_{!*}^{\mathfrak{F}}[d_X]$ establishes an equivalence of categories between $Loc(S_0)$ and the full subcategory of complexes $\mathcal{F}^\bullet$ in $\mathcal{D}^b_{\mathfrak{X}-c}(X)$ verifying the following conditions:

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < -d_X$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > -d_X$ *and* $\mathcal{H}^{-d_X}(\mathcal{F}^\bullet)|_{S_0} \in Loc(S_0)$
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$
- *(S')* $\mathcal{H}^m(\mathbb{D}_X\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$.

Intersection homology complexes

From now on we will denote by X a complex quasi-projective variety and by $\mathfrak{X}$ and $\mathfrak{F}$ an algebraic Whitney stratification and the associated filtration respectively. For a stratum S we will denote by d_S the complex dimension of the stratum.

Theorem

Using the above notations the intermediate extension functor $\iota^{\mathfrak{F}}_{!*}[d_X]$ establishes an equivalence of categories between $Loc(S_0)$ and the full subcategory of complexes $\mathcal{F}^\bullet$ in $\mathcal{D}^b_{\mathfrak{X}-c}(X)$ verifying the following conditions:

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < -d_X$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > -d_X$ *and* $\mathcal{H}^{-d_X}(\mathcal{F}^\bullet)|_{S_0} \in Loc(S_0)$
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$
- *(S')* $\mathcal{H}^m(\mathbb{D}_X\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$.

Intersection homology complexes

From now on we will denote by X a complex quasi-projective variety and by $\mathfrak{X}$ and $\mathfrak{F}$ an algebraic Whitney stratification and the associated filtration respectively. For a stratum S we will denote by d_S the complex dimension of the stratum.

Theorem

Using the above notations the intermediate extension functor $\iota^{\mathfrak{F}}_{!*}[d_X]$ establishes an equivalence of categories between $Loc(S_0)$ and the full subcategory of complexes $\mathcal{F}^\bullet$ in $\mathcal{D}^b_{\mathfrak{X}-c}(X)$ verifying the following conditions:

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < -d_X$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > -d_X$ *and* $\mathcal{H}^{-d_X}(\mathcal{F}^\bullet)|_{S_0} \in Loc(S_0)$
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$
- *(S')* $\mathcal{H}^m(\mathbb{D}_X\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S.$

Intersection homology complexes

From now on we will denote by X a complex quasi-projective variety and by $\mathfrak{X}$ and $\mathfrak{F}$ an algebraic Whitney stratification and the associated filtration respectively. For a stratum S we will denote by d_S the complex dimension of the stratum.

Theorem

Using the above notations the intermediate extension functor $\iota_{!*}^{\mathfrak{F}}[d_X]$ establishes an equivalence of categories between $Loc(S_0)$ and the full subcategory of complexes $\mathcal{F}^\bullet$ in $\mathcal{D}^b_{\mathfrak{X}-c}(X)$ verifying the following conditions:

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < -d_X$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > -d_X$ *and* $\mathcal{H}^{-d_X}(\mathcal{F}^\bullet)|_{S_0} \in Loc(S_0)$
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$
- *(S')* $\mathcal{H}^m(\mathbb{D}_X\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S.$

Intersection homology complexes

From now on we will denote by X a complex quasi-projective variety and by $\mathfrak{X}$ and $\mathfrak{F}$ an algebraic Whitney stratification and the associated filtration respectively. For a stratum S we will denote by d_S the complex dimension of the stratum.

Theorem

Using the above notations the intermediate extension functor $\iota^{\mathfrak{F}}_{!*}[d_X]$ establishes an equivalence of categories between $Loc(S_0)$ and the full subcategory of complexes $\mathcal{F}^\bullet$ in $\mathcal{D}^b_{\mathfrak{X}-c}(X)$ verifying the following conditions:

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < -d_X$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > -d_X$ *and* $\mathcal{H}^{-d_X}(\mathcal{F}^\bullet)|_{S_0} \in Loc(S_0)$
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$
- *(S')* $\mathcal{H}^m(\mathbb{D}_X\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$.

Intersection homology complexes

We can depict the degrees/strata where we can have non-zero cohomology for $\mathcal{F}^\bullet, \mathbb{D}_X\mathcal{F}^\bullet$ as in the theorem. Namely

	$-d_X-1$	$-d_X$	$-d_X+1$	$-d_X+2$	$-d_X+3$	$-d_X+4$	....
$\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_0}$	0	$\bullet$	0	0	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_1}$	0	$\bullet$	0	0	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_2}$	0	$\bullet$	$\bullet$	0	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_3}$	0	$\bullet$	$\bullet$	$\bullet$	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_4}$	0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_5}$	0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	0

Intersection homology complexes

We will denote by $IC^{\bullet}_{\mathfrak{X}}(X;\mathcal{L})$ the complex $\iota^{\mathfrak{F}}_{!*}(\mathcal{L})$ where $\mathcal{L}$ is a local system on S_0. This is called the *intersection homology complex* of X with coefficients in $\mathcal{L}$.

We can now deduce some corollaries

Corollary

Let $\mathfrak{X} < \mathfrak{Y}$ and let S_0 respectively T_0 be codimension one strata and $\iota : T_0 \hookrightarrow S_0$ be the inclusion. Then we have $\mathfrak{IC}_{\mathfrak{X}}(X) \subseteq \mathfrak{IC}_{\mathfrak{Y}}(X)$ and moreover the following diagram commutes:

$$\begin{array}{ccc} Loc(S_0) & \xrightarrow{\iota^{\mathfrak{X}}_{!*}} & \mathfrak{IC}_{\mathfrak{X}}(X) \\ \iota^{-1}\downarrow & & \downarrow \\ Loc(T_0) & \xrightarrow{\iota^{\mathfrak{Y}}_{!*}} & \mathfrak{IC}_{\mathfrak{Y}}(X) \end{array}$$

Intersection homology complexes

We will denote by $IC^{\bullet}_{\mathfrak{X}}(X;\mathcal{L})$ the complex $\iota^{\mathfrak{F}}_{!*}(\mathcal{L})$ where $\mathcal{L}$ is a local system on S_0. This is called the *intersection homology complex* of X with coefficients in $\mathcal{L}$.

We can now deduce some corollaries

Corollary

Let $\mathfrak{X} < \mathfrak{Y}$ and let S_0 respectively T_0 be codimension one strata and $\iota : T_0 \hookrightarrow S_0$ be the inclusion. Then we have $\mathfrak{IC}_{\mathfrak{X}}(X) \subseteq \mathfrak{IC}_{\mathfrak{Y}}(X)$ and moreover the following diagram commutes:

$$\begin{array}{ccc} Loc(S_0) & \xrightarrow{\iota^{\mathfrak{X}}_{!*}} & \mathfrak{IC}_{\mathfrak{X}}(X) \\ {\scriptstyle \iota^{-1}}\downarrow & & \downarrow \\ Loc(T_0) & \xrightarrow{\iota^{\mathfrak{Y}}_{!*}} & \mathfrak{IC}_{\mathfrak{Y}}(X) \end{array}$$

Intersection homology complexes

Corollary

Let X be a complex algebraic variety.

1. *For any algebraic Whitney stratification $\mathfrak{X}$ we have that $\mathfrak{IC}_{\mathfrak{X}}(X)$ is abelian and stable under the action of the Verdier duality $\mathbb{D}_X$.*
2. *For any refinement of algebraic Whitney stratifications $\mathfrak{X} < \mathfrak{Y}$ we have that the inclusion $\mathfrak{IC}_{\mathfrak{X}}(X) \subseteq \mathfrak{IC}_{\mathfrak{Y}}(X)$ is faithfully full and exact.*
3. *For any local system $\mathcal{L} \in Loc(S_0)$ we have a canonical isomorphism*

$$\mathbb{D}_X(IC^{\bullet}_{\mathfrak{X}}(X;\mathcal{L})) \simeq IC^{\bullet}_{\mathfrak{X}}(X;\mathcal{L}^{\vee})$$

Intersection homology complexes

Corollary

Let X be a complex algebraic variety.

1. *For any algebraic Whitney stratification $\mathfrak{X}$ we have that $\mathfrak{I}\mathcal{C}_{\mathfrak{X}}(X)$ is abelian and stable under the action of the Verdier duality $\mathbb{D}_X$.*
2. *For any refinement of algebraic Whitney stratifications $\mathfrak{X} < \mathfrak{Y}$ we have that the inclusion $\mathfrak{I}\mathcal{C}_{\mathfrak{X}}(X) \subseteq \mathfrak{I}\mathcal{C}_{\mathfrak{Y}}(X)$ is faithfully full and exact.*
3. *For any local system $\mathcal{L} \in Loc(S_0)$ we have a canonical isomorphism*
$$\mathbb{D}_X(IC^{\bullet}_{\mathfrak{X}}(X;\mathcal{L})) \simeq IC^{\bullet}_{\mathfrak{X}}(X;\mathcal{L}^{\vee})$$

Intersection homology complexes

Corollary

Let X be a complex algebraic variety.

1. *For any algebraic Whitney stratification $\mathfrak{X}$ we have that $\mathfrak{IC}_{\mathfrak{X}}(X)$ is abelian and stable under the action of the Verdier duality $\mathbb{D}_X$.*
2. *For any refinement of algebraic Whitney stratifications $\mathfrak{X} < \mathfrak{Y}$ we have that the inclusion $\mathfrak{IC}_{\mathfrak{X}}(X) \subseteq \mathfrak{IC}_{\mathfrak{Y}}(X)$ is faithfully full and exact.*
3. *For any local system $\mathcal{L} \in Loc(S_0)$ we have a canonical isomorphism*

$$\mathbb{D}_X(IC^{\bullet}_{\mathfrak{X}}(X;\mathcal{L})) \simeq IC^{\bullet}_{\mathfrak{X}}(X;\mathcal{L}^{\vee})$$

Corollary

Let X be a complex algebraic variety.

1. *For any algebraic Whitney stratification $\mathfrak{X}$ we have that $\mathfrak{IC}_{\mathfrak{X}}(X)$ is abelian and stable under the action of the Verdier duality $\mathbb{D}_X$.*
2. *For any refinement of algebraic Whitney stratifications $\mathfrak{X} < \mathfrak{Y}$ we have that the inclusion $\mathfrak{IC}_{\mathfrak{X}}(X) \subseteq \mathfrak{IC}_{\mathfrak{Y}}(X)$ is faithfully full and exact.*
3. *For any local system $\mathcal{L} \in Loc(S_0)$ we have a canonical isomorphism*

$$\mathbb{D}_X(IC^\bullet_{\mathfrak{X}}(X;\mathcal{L})) \simeq IC^\bullet_{\mathfrak{X}}(X;\mathcal{L}^\vee)$$

Deligne-Goresky-MacPherson Complexes

Let Z be a closed subset of X which is a union of strata.

For any local system $\mathcal{L}$ on an open dense Zarisky subset of Z we have the complex $IC^\bullet_{\mathfrak{X}}(Z,\mathcal{L})$ in $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$ and we can consider its pushforward $i_{Z*}(IC^\bullet_{\mathfrak{X}}(Z,\mathcal{L})) \in \mathcal{D}^b_{\mathfrak{X}-c}(X)$.

The following table illustrates/resumes the properties of the above complex:

	$-d_X$	$-d_X+1$	...	$-d_Z$	$-d_Z+1$	$-d_Z+2$	$-d_Z+3$
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_0}$	0	0	...	0	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_1}$	0	0	...	0	0	0	0
...	...	...	...	...	...	...	...
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z}}$	0	0	0	•	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+1}}$	0	0	0	•	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+2}}$	0	0	0	•	•	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+3}}$	0	0	0	•	•	•	0

where $\mathcal{F}^\bullet = i_{Z*}IC^\bullet_{\mathfrak{X}}(Z,\mathcal{L})[d_Z]$ or its Verdier dual.

Deligne-Goresky-MacPherson Complexes

Let Z be a closed subset of X which is a union of strata.
For any local system $\mathcal{L}$ on an open dense Zarisky subset of Z we have the complex $IC^{\bullet}_{\mathfrak{X}}(Z, \mathcal{L})$ in $\mathcal{D}^{b}_{\mathfrak{X}-c}(Z)$ and we can consider its pushforward $i_{Z*}(IC^{\bullet}_{\mathfrak{X}}(Z, \mathcal{L})) \in \mathcal{D}^{b}_{\mathfrak{X}-c}(X)$.
The following table illustrates/resumes the properties of the above complex:

	$-d_X$	$-d_X+1$	...	$-d_Z$	$-d_Z+1$	$-d_Z+2$	$-d_Z+3$
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_0}$	0	0	...	0	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_1}$	0	0	...	0	0	0	0
...	...	...	...	...	...	...	...
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z}}$	0	0	0	$\bullet$	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+1}}$	0	0	0	$\bullet$	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+2}}$	0	0	0	$\bullet$	$\bullet$	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+3}}$	0	0	0	$\bullet$	$\bullet$	$\bullet$	0

where $\mathcal{F}^\bullet = i_{Z*}IC^{\bullet}_{\mathfrak{X}}(Z, \mathcal{L})[d_Z]$ or its Verdier dual.

Deligne-Goresky-MacPherson Complexes

Let Z be a closed subset of X which is a union of strata.
For any local system $\mathcal{L}$ on an open dense Zarisky subset of Z we have the complex $IC^{\bullet}_{\mathfrak{X}}(Z,\mathcal{L})$ in $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$ and we can consider its pushforward $i_{Z*}(IC^{\bullet}_{\mathfrak{X}}(Z,\mathcal{L})) \in \mathcal{D}^b_{\mathfrak{X}-c}(X)$.
The following table illustrates/resumes the properties of the above complex:

	$-d_X$	$-d_X+1$	...	$-d_Z$	$-d_Z+1$	$-d_Z+2$	$-d_Z+3$
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_0}$	0	0	...	0	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_1}$	0	0	...	0	0	0	0
...	...	...	...	...	...	...	...
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z}}$	0	0	0	$\bullet$	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+1}}$	0	0	0	$\bullet$	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+2}}$	0	0	0	$\bullet$	$\bullet$	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+3}}$	0	0	0	$\bullet$	$\bullet$	$\bullet$	0

where $\mathcal{F}^\bullet = i_{Z*}IC^{\bullet}_{\mathfrak{X}}(Z,\mathcal{L})[d_Z]$ or its Verdier dual.

Deligne-Goresky-MacPherson Complexes

Definition

A complex $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(X)$ is called a *DGM-complex relative to* $\mathfrak{X}$ if there exists some closed irreducible subvariety $Z \subseteq X$ which is a union of strata from $\mathfrak{X}$ and an **irreducible** local system on a non-singular dense open subset of Z such that $\mathcal{F}^\bullet \simeq i_{Z*}(IC^\bullet_{\mathfrak{X}}(Z, \mathcal{L}))[d_Z]$. We denote them by $DGM_{\mathfrak{X}}(X)$.

$\mathfrak{X}$-perverse sheaves

Definition

Let X be a complex algebraic variety and $\mathfrak{X}$ a stratification. A complex of sheaves $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(X)$ is called $\mathfrak{X}$-perverse if for each stratum $S \in \mathfrak{X}$ we have:

- (S) $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$
- (S) $\mathcal{H}^m(\mathbb{D}_X(\mathcal{F}^\bullet))|_S = 0, \forall m > -d_S$.

The full subcategory of $\mathfrak{X}$-perverse sheaves of $\mathcal{D}^b_{\mathfrak{X}-c}$ is denoted $\mathcal{P}erv_{\mathfrak{X}}(X)$.

Remark

From the previous discussion we deduce that if $Z \subseteq X$ is a closed set which is a union of strata then

$$i_{Z*}(\mathfrak{IC}_{\mathfrak{X}}(Z))[d_Z] \subseteq \mathcal{P}erv_{\mathfrak{X}}(X).$$

$\mathfrak{X}$-perverse sheaves

Definition

Let X be a complex algebraic variety and $\mathfrak{X}$ a stratification. A complex of sheaves $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(X)$ is called $\mathfrak{X}$-perverse if for each stratum $S \in \mathfrak{X}$ we have:

- (S) $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0, \forall m > -d_S$
- (S) $\mathcal{H}^m(\mathbb{D}_X(\mathcal{F}^\bullet))|_S = 0, \forall m > -d_S$.

The full subcategory of $\mathfrak{X}$-perverse sheaves of $\mathcal{D}^b_{\mathfrak{X}-c}$ is denoted $\mathcal{P}erv_{\mathfrak{X}}(X)$.

Remark

From the previous discussion we deduce that if $Z \subseteq X$ is a closed set which is a union of strata then

$$i_{Z*}(\mathfrak{IC}_{\mathfrak{X}}(Z))[d_Z] \subseteq \mathcal{P}erv_{\mathfrak{X}}(X).$$

$\mathfrak{X}$-perverse sheaves

Definition

Let X be a complex algebraic variety and $\mathfrak{X}$ a stratification. A complex of sheaves $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(X)$ is called $\mathfrak{X}$-perverse if for each stratum $S \in \mathfrak{X}$ we have:

- (S) $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0, \ \forall m > -d_S$
- (S) $\mathcal{H}^m(\mathbb{D}_X(\mathcal{F}^\bullet))|_S = 0, \ \forall m > -d_S$.

The full subcategory of $\mathfrak{X}$-perverse sheaves of $\mathcal{D}^b_{\mathfrak{X}-c}$ is denoted $\mathcal{P}erv_{\mathfrak{X}}(X)$.

Remark

From the previous discussion we deduce that if $Z \subseteq X$ is a closed set which is a union of strata then

$$i_{Z*}(\mathfrak{I}\mathcal{C}_{\mathfrak{X}}(Z))[d_Z] \subseteq \mathcal{P}erv_{\mathfrak{X}}(X).$$

$\mathfrak{X}$-perverse sheaves

From the support and cosupport conditions we can prove that a perverse sheaf has the cohomology concentrated in degrees $[-d_X, 0]$. So we have the following picture:

	$-d_X-1$	$-d_X$	$-d_X+1$	$-d_X+2$	$-d_X+3$	$-d_X+4$	....
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_0}$	0	•	0	0	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_1}$	0	•	•	0	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_2}$	0	•	•	•	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_3}$	0	•	•	•	•	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_4}$	0	•	•	•	•	•	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_5}$	0	•	•	•	•	•	•

$\mathfrak{X}$-perverse sheaves

From the support and cosupport conditions we can prove that a perverse sheaf has the cohomology concentrated in degrees $[-d_X, 0]$. So we have the following picture:

	$-d_X - 1$	$-d_X$	$-d_X + 1$	$-d_X + 2$	$-d_X + 3$	$-d_X + 4$	....
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_0}$	0	$\bullet$	0	0	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_1}$	0	$\bullet$	$\bullet$	0	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_2}$	0	$\bullet$	$\bullet$	$\bullet$	0	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_3}$	0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	0	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_4}$	0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	0
$\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_5}$	0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$

$\mathfrak{X}$-perverse sheaves

Proposition

Let Z be a locally closed subset of a variety X which is a union of strata of $\mathfrak{X}$.

1. If Z is closed then $i_{Z*}(\mathcal{P}erv_{\mathfrak{X}}(Z)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$
2. If Z is open then $i_Z^{-1}(\mathcal{P}erv_{\mathfrak{X}}(X)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(Z)$
3. Let $Z = \coprod_i S_i$ be a union of open strata of $\mathfrak{X}$. Then for any $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ we have $i_Z^{-1}\mathcal{F}^\bullet = \oplus_i \mathcal{L}_i[d_{S_i}]$ where $\mathcal{L}_i$ are local systems on S_i.
4. If $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ has the property that $|\mathcal{F}^\bullet| \subseteq Z$ then $i_Z^{-1}\mathcal{F}^\bullet = i_Z^!\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(Z)$
5. If Z is an **open affine non-singular** subvariety of X then $Ri_{Z*}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$ and $i_{Z!}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$.

$\mathfrak{X}$-perverse sheaves

Proposition

Let Z be a locally closed subset of a variety X which is a union of strata of $\mathfrak{X}$.

1. If Z is closed then $i_{Z*}(\mathcal{P}erv_{\mathfrak{X}}(Z)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$
2. If Z is open then $i_Z^{-1}(\mathcal{P}erv_{\mathfrak{X}}(X)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(Z)$
3. Let $Z = \coprod_i S_i$ be a union of open strata of $\mathfrak{X}$. Then for any $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ we have $i_Z^{-1}\mathcal{F}^\bullet = \oplus_i \mathcal{L}_i[d_{S_i}]$ where $\mathcal{L}_i$ are local systems on S_i.
4. If $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ has the property that $|\mathcal{F}^\bullet| \subseteq Z$ then $i_Z^{-1}\mathcal{F}^\bullet = i_Z^! \mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(Z)$
5. If Z is an **open affine non-singular** subvariety of X then $Ri_{Z*}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$ and $i_{Z!}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$.

$\mathfrak{X}$-perverse sheaves

Proposition

Let Z be a locally closed subset of a variety X which is a union of strata of $\mathfrak{X}$.

1. If Z is closed then $i_{Z*}(\mathcal{P}erv_{\mathfrak{X}}(Z)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$
2. If Z is open then $i_Z^{-1}(\mathcal{P}erv_{\mathfrak{X}}(X)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(Z)$
3. Let $Z = \coprod_i S_i$ be a union of open strata of $\mathfrak{X}$. Then for any $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ we have $i_Z^{-1}\mathcal{F}^\bullet = \oplus_i \mathcal{L}_i[d_{S_i}]$ where $\mathcal{L}_i$ are local systems on S_i.
4. If $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ has the property that $|\mathcal{F}^\bullet| \subseteq Z$ then $i_Z^{-1}\mathcal{F}^\bullet = i_Z^!\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(Z)$
5. If Z is an **open affine non-singular** subvariety of X then $Ri_{Z*}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$ and $i_{Z!}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$.

$\mathfrak{X}$-perverse sheaves

Proposition

Let Z be a locally closed subset of a variety X which is a union of strata of $\mathfrak{X}$.

1. If Z is closed then $i_{Z*}(\mathcal{P}erv_{\mathfrak{X}}(Z)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$
2. If Z is open then $i_Z^{-1}(\mathcal{P}erv_{\mathfrak{X}}(X)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(Z)$
3. Let $Z = \coprod_i S_i$ be a union of open strata of $\mathfrak{X}$. Then for any $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ we have $i_Z^{-1}\mathcal{F}^\bullet = \oplus_i \mathcal{L}_i[d_{S_i}]$ where $\mathcal{L}_i$ are local systems on S_i.
4. If $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ has the property that $|\mathcal{F}^\bullet| \subseteq Z$ then $i_Z^{-1}\mathcal{F}^\bullet = i_Z^!\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(Z)$
5. If Z is an **open affine non-singular** subvariety of X then $Ri_{Z*}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$ and $i_{Z!}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$.

$\mathfrak{X}$-perverse sheaves

Proposition

Let Z be a locally closed subset of a variety X which is a union of strata of $\mathfrak{X}$.

1. If Z is closed then $i_{Z*}(\mathcal{P}erv_{\mathfrak{X}}(Z)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$
2. If Z is open then $i_Z^{-1}(\mathcal{P}erv_{\mathfrak{X}}(X)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(Z)$
3. Let $Z = \coprod_i S_i$ be a union of open strata of $\mathfrak{X}$. Then for any $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ we have $i_Z^{-1}\mathcal{F}^\bullet = \oplus_i \mathcal{L}_i[d_{S_i}]$ where $\mathcal{L}_i$ are local systems on S_i.
4. If $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ has the property that $|\mathcal{F}^\bullet| \subseteq Z$ then $i_Z^{-1}\mathcal{F}^\bullet = i_Z^!\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(Z)$
5. If Z is an **open affine non-singular** subvariety of X then $Ri_{Z*}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$ and $i_{Z!}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$.

$\mathfrak{X}$-perverse sheaves

Proposition

Let Z be a locally closed subset of a variety X which is a union of strata of $\mathfrak{X}$.

1. If Z is closed then $i_{Z*}(\mathcal{P}erv_{\mathfrak{X}}(Z)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$
2. If Z is open then $i_Z^{-1}(\mathcal{P}erv_{\mathfrak{X}}(X)) \subseteq \mathcal{P}erv_{\mathfrak{X}}(Z)$
3. Let $Z = \coprod_i S_i$ be a union of open strata of $\mathfrak{X}$. Then for any $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ we have $i_Z^{-1}\mathcal{F}^\bullet = \oplus_i \mathcal{L}_i[d_{S_i}]$ where $\mathcal{L}_i$ are local systems on S_i.
4. If $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(X)$ has the property that $|\mathcal{F}^\bullet| \subseteq Z$ then $i_Z^{-1}\mathcal{F}^\bullet = i_Z^!\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathfrak{X}}(Z)$
5. If Z is an **open affine non-singular** subvariety of X then $Ri_{Z*}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$ and $i_{Z!}(Loc(Z)[d_Z]) \subseteq \mathcal{P}erv_{\mathfrak{X}}(X)$.

$\mathfrak{X}$-perverse sheaves and perverse t-structures

Definition

For each $Z \subseteq X$ a locally closed subset that is a union of strata we define the following subcategories of $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$:

- $\mathcal{D}^{\leq 0}_{\mathfrak{X},Z} = \{\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(Z) : \mathcal{H}^m(\mathcal{F}^\bullet)|_T = 0, \forall m > -d_T, \forall T \in \mathfrak{X}|_Z\}$
- $\mathcal{D}^{\geq 0}_{\mathfrak{X},Z} = \{\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(Z) : \mathcal{H}^m(\mathbb{D}_Z\mathcal{F}^\bullet)|_T = 0, \forall m > -d_T, \forall T \in \mathfrak{X}|_Z\}$

Proposition

The pair $(\mathcal{D}^{\leq 0}_{\mathfrak{X},Z}, \mathcal{D}^{\geq 0}_{\mathfrak{X},Z})$ is a t-structure on $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$ for any Z as above.

Remark

From the above discussions we have that

$$\mathcal{P}erv_{\mathfrak{X}}(X) = \mathcal{D}^{\leq 0}_{\mathfrak{X},Z} \cap \mathcal{D}^{\geq 0}_{\mathfrak{X},Z}$$

$\mathfrak{X}$-perverse sheaves and perverse t-structures

Definition

For each $Z \subseteq X$ a locally closed subset that is a union of strata we define the following subcategories of $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$:

- $\mathcal{D}^{\leq 0}_{\mathfrak{X},Z} = \{\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(Z) \,:\, \mathcal{H}^m(\mathcal{F}^\bullet)|_T = 0,\ \forall m > -d_T,\ \forall T \in \mathfrak{X}|_Z\}$
- $\mathcal{D}^{\geq 0}_{\mathfrak{X},Z} = \{\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(Z) \,:\, \mathcal{H}^m(\mathbb{D}_Z\mathcal{F}^\bullet)|_T = 0, \forall m > -d_T,\ \forall T \in \mathfrak{X}|_Z\}$

Proposition

The pair $(\mathcal{D}^{\leq 0}_{\mathfrak{X},Z}, \mathcal{D}^{\geq 0}_{\mathfrak{X},Z})$ is a t-structure on $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$ for any Z as above.

Remark

From the above discussions we have that

$$\mathcal{P}erv_{\mathfrak{X}}(X) = \mathcal{D}^{\leq 0}_{\mathfrak{X},Z} \cap \mathcal{D}^{\geq 0}_{\mathfrak{X},Z}$$

$\mathfrak{X}$-perverse sheaves and perverse t-structures

Definition

For each $Z \subseteq X$ a locally closed subset that is a union of strata we define the following subcategories of $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$:

- $\mathcal{D}^{\leq 0}_{\mathfrak{X},Z} = \{\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(Z) \,:\, \mathcal{H}^m(\mathcal{F}^\bullet)|_T = 0,\ \forall m > -d_T,$ $\forall T \in \mathfrak{X}|_Z\}$
- $\mathcal{D}^{\geq 0}_{\mathfrak{X},Z} = \{\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(Z) \,:\, \mathcal{H}^m(\mathbb{D}_Z\mathcal{F}^\bullet)|_T = 0, \forall m > -d_T,$ $\forall T \in \mathfrak{X}|_Z\}$

Proposition

The pair $(\mathcal{D}^{\leq 0}_{\mathfrak{X},Z}, \mathcal{D}^{\geq 0}_{\mathfrak{X},Z})$ is a t-structure on $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$ for any Z as above.

Remark

From the above discussions we have that

$$\mathcal{P}erv_{\mathfrak{X}}(X) = \mathcal{D}^{\leq 0}_{\mathfrak{X},Z} \cap \mathcal{D}^{\geq 0}_{\mathfrak{X},Z}$$

$\mathfrak{X}$-perverse sheaves and perverse t-structures

Definition

For each $Z \subseteq X$ a locally closed subset that is a union of strata we define the following subcategories of $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$:

- $\mathcal{D}^{\leq 0}_{\mathfrak{X},Z} = \{\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(Z) \,:\, \mathcal{H}^m(\mathcal{F}^\bullet)|_T = 0, \forall m > -d_T, \forall T \in \mathfrak{X}|_Z\}$
- $\mathcal{D}^{\geq 0}_{\mathfrak{X},Z} = \{\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(Z) \,:\, \mathcal{H}^m(\mathbb{D}_Z\mathcal{F}^\bullet)|_T = 0, \forall m > -d_T, \forall T \in \mathfrak{X}|_Z\}$

Proposition

The pair $(\mathcal{D}^{\leq 0}_{\mathfrak{X},Z}, \mathcal{D}^{\geq 0}_{\mathfrak{X},Z})$ is a t-structure on $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$ for any Z as above.

Remark

From the above discussions we have that

$$\mathcal{P}erv_{\mathfrak{X}}(X) = \mathcal{D}^{\leq 0}_{\mathfrak{X},Z} \cap \mathcal{D}^{\geq 0}_{\mathfrak{X},Z}$$

$\mathfrak{X}$-perverse sheaves and perverse t-structures

Definition

For each $Z \subseteq X$ a locally closed subset that is a union of strata we define the following subcategories of $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$:

- $\mathcal{D}^{\leq 0}_{\mathfrak{X},Z} = \{\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(Z) \ : \ \mathcal{H}^m(\mathcal{F}^\bullet)|_T = 0, \forall m > -d_T, \forall T \in \mathfrak{X}|_Z\}$
- $\mathcal{D}^{\geq 0}_{\mathfrak{X},Z} = \{\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(Z) \ : \ \mathcal{H}^m(\mathbb{D}_Z\mathcal{F}^\bullet)|_T = 0, \forall m > -d_T, \forall T \in \mathfrak{X}|_Z\}$

Proposition

The pair $(\mathcal{D}^{\leq 0}_{\mathfrak{X},Z}, \mathcal{D}^{\geq 0}_{\mathfrak{X},Z})$ is a t-structure on $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$ for any Z as above.

Remark

From the above discussions we have that

$$\mathcal{P}erv_{\mathfrak{X}}(X) = \mathcal{D}^{\leq 0}_{\mathfrak{X},Z} \cap \mathcal{D}^{\geq 0}_{\mathfrak{X},Z}$$

Structure theorems for $\mathfrak{X}$-perverse sheaves

Theorem (BBD)

For any variety X and any algebraic Whitney stratification $\mathfrak{X}$ the full subcategory $\mathcal{P}erv_{\mathfrak{X}}(X)$ of $\mathcal{D}^b_{\mathfrak{X}-c}(X)$ is an *abelian*, *admissible category* that is *stable by extensions and by Verdier duality*.

Structure theorems for $\mathfrak{X}$-perverse sheaves

Recall that a DGM-complex relative to a stratification $\mathfrak{X}$ is a complex isomorphic to $i_{Z*}IC^{\bullet}_{\mathfrak{X}}(Z, \mathcal{L})$ where Z is an irreducible closed subvariety of X union of strata and $\mathcal{L}$ is an irreducible local system on a dense open subset of Z. We have the following theorem

Theorem

The DGM-complexes are simple objects in the category $\mathcal{P}erv_{\mathfrak{X}}(X)$.

Recall that a DGM-complex relative to a stratification $\mathfrak{X}$ is a complex isomorphic to $i_{Z*}IC^{\bullet}_{\mathfrak{X}}(Z,\mathcal{L})$ where Z is an irreducible closed subvariety of X union of strata and $\mathcal{L}$ is an irreducible local system on a dense open subset of Z. We have the following theorem

Theorem

The DGM-complexes are simple objects in the category $\mathcal{P}erv_{\mathfrak{X}}(X)$.

Structure theorems for perverse sheaves

Definition

The category of *perverse sheaves* on X is the full subcategory of $\mathcal{D}^b(X)$ consisting of objects that are $\mathfrak{X}$-perverse for some algebraic Whitney stratification $\mathfrak{X}$. We denote it by $\mathcal{P}erv(X)$. In other words

$$\mathcal{P}erv(X) = \varinjlim_{\mathfrak{X}} \mathcal{P}erv_{\mathfrak{X}}(X).$$

Definition

A *DGM*-complex on X is a perverse sheaf that is a *DGM*-complex relative to $\mathfrak{X}$ for an algebraic Whitney stratification. Again, we can express this by

$$DGM(X) = \varinjlim_{\mathfrak{X}} DGM_{\mathfrak{X}}(X)$$

Structure theorems for perverse sheaves

Definition

The category of *perverse sheaves* on X is the full subcategory of $\mathcal{D}^b(X)$ consisting of objects that are $\mathfrak{X}$-perverse for some algebraic Whitney stratification $\mathfrak{X}$. We denote it by $\mathcal{P}erv(X)$. In other words

$$\mathcal{P}erv(X) = \varinjlim_{\mathfrak{X}} \mathcal{P}erv_{\mathfrak{X}}(X).$$

Definition

A *DGM*-complex on X is a perverse sheaf that is a *DGM*-complex relative to $\mathfrak{X}$ for an algebraic Whitney stratification. Again, we can express this by

$$DGM(X) = \varinjlim_{\mathfrak{X}} DGM_{\mathfrak{X}}(X)$$

Structure theorems for perverse sheaves

Definition

The category of *perverse sheaves* on X is the full subcategory of $\mathcal{D}^b(X)$ consisting of objects that are $\mathfrak{X}$-perverse for some algebraic Whitney stratification $\mathfrak{X}$. We denote it by $\mathcal{P}erv(X)$. In other words

$$\mathcal{P}erv(X) = \varinjlim_{\mathfrak{X}} \mathcal{P}erv_{\mathfrak{X}}(X).$$

Definition

A *DGM*-complex on X is a perverse sheaf that is a *DGM*-complex relative to $\mathfrak{X}$ for an algebraic Whitney stratification. Again, we can express this by

$$DGM(X) = \varinjlim_{\mathfrak{X}} DGM_{\mathfrak{X}}(X)$$

Structure theorems for perverse sheaves

Definition

The category of *perverse sheaves* on X is the full subcategory of $\mathcal{D}^b(X)$ consisting of objects that are $\mathfrak{X}$-perverse for some algebraic Whitney stratification $\mathfrak{X}$. We denote it by $\mathcal{P}erv(X)$. In other words

$$\mathcal{P}erv(X) = \varinjlim_{\mathfrak{X}} \mathcal{P}erv_{\mathfrak{X}}(X).$$

Definition

A *DGM*-complex on X is a perverse sheaf that is a *DGM*-complex relative to $\mathfrak{X}$ for an algebraic Whitney stratification. Again, we can express this by

$$DGM(X) = \varinjlim_{\mathfrak{X}} DGM_{\mathfrak{X}}(X)$$

Structure theorems for perverse sheaves

Theorem (BBD)

Let X be a variety. We have

- The category $\mathcal{P}erv(X)$ is a full subcategory of $\mathcal{D}^b_c(X)$ that is abelian, stable by extensions and by Verdier duality.
- The simple objects of $\mathcal{P}erv(X)$ are precisely the *DGM*-complexes.
- All the objects of $\mathcal{P}erv(X)$ are finite successive extensions of simple objects: the category of perverse sheaves is artinian and noetherian.

Structure theorems for perverse sheaves

Theorem (BBD)

Let X be a variety. We have

- The category $\mathcal{P}erv(X)$ is a full subcategory of $\mathcal{D}_c^b(X)$ that is abelian, stable by extensions and by Verdier duality.
- The simple objects of $\mathcal{P}erv(X)$ are precisely the *DGM*-complexes.
- All the objects of $\mathcal{P}erv(X)$ are finite successive extensions of simple objects: the category of perverse sheaves is artinian and noetherian.

Structure theorems for perverse sheaves

Theorem (BBD)

Let X be a variety. We have

- The category $\mathcal{P}erv(X)$ is a full subcategory of $\mathcal{D}_c^b(X)$ that is abelian, stable by extensions and by Verdier duality.
- The simple objects of $\mathcal{P}erv(X)$ are precisely the *DGM*-complexes.
- All the objects of $\mathcal{P}erv(X)$ are finite successive extensions of simple objects: the category of perverse sheaves is artinian and noetherian.

Structure theorems for perverse sheaves

Theorem (BBD)

Let X be a variety. We have

- The category $\mathcal{P}erv(X)$ is a full subcategory of $\mathcal{D}_c^b(X)$ that is abelian, stable by extensions and by Verdier duality.
- The simple objects of $\mathcal{P}erv(X)$ are precisely the *DGM*-complexes.
- All the objects of $\mathcal{P}erv(X)$ are finite successive extensions of simple objects: the category of perverse sheaves is artinian and noetherian.

Stratified morphisms

Proposition

Let $(X, \mathfrak{X}), (Y, \mathfrak{Y})$ be algebraic varieties with Whitney stratifications and $f : X \to Y$ a stratified map. Then the following hold:

$$f^{-1}\mathcal{D}^b_{\mathfrak{Y}-c}(Y) \subseteq \mathcal{D}^b_{\mathfrak{X}-c}, \quad f^!\mathcal{D}^b_{\mathfrak{Y}-c}(Y) \subseteq \mathcal{D}^b_{\mathfrak{X}-c}$$

$$Rf_*\mathcal{D}^b_{\mathfrak{X}-c}(X) \subseteq \mathcal{D}^b_{\mathfrak{Y}-c}, \quad Rf_!\mathcal{D}^b_{\mathfrak{X}-c}(Y) \subseteq \mathcal{D}^b_{\mathfrak{Y}-c}$$

Therefore, we have all the above stability properties for $\mathcal{D}^b_c(X)$ and $\mathcal{D}^b_c(Y)$.

The Decomposition Theorem

Theorem

Let X, Y be two complex algebraic varieties and $f : X \to Y$ a **proper** algebraic map. For any simple* perverse sheaf $i_{Z*}(IC^\bullet(Z, \mathcal{L}))$ on X there exist a finite number of irreducible closed sets $Z_i \subseteq Y$, irreducible local systems $\mathcal{L}_i$ on open subsets of Z_i and integers c_i such that

$$Rf_*(i_{Z*}IC^\bullet(Z, \mathcal{L})[d_Z]) = \bigoplus_i i_{Z_i*}(IC^\bullet(Z_i, \mathcal{L}_i))[c_i].$$

Remark

If the map is stratified with respect to the stratifications $(\mathfrak{X}, \mathfrak{Y})$ then we can choose Z_i to be strata from $\mathfrak{Y}$.

The Decomposition Theorem

Theorem

Let X, Y be two complex algebraic varieties and $f : X \to Y$ a **proper** algebraic map. For any simple* perverse sheaf $i_{Z*}(IC^\bullet(Z, \mathcal{L}))$ on X there exist a finite number of irreducible closed sets $Z_i \subseteq Y$, irreducible local systems $\mathcal{L}_i$ on open subsets of Z_i and integers c_i such that

$$Rf_*(i_{Z*}IC^\bullet(Z, \mathcal{L})[d_Z]) = \bigoplus_i i_{Z_i*}(IC^\bullet(Z_i, \mathcal{L}_i))[c_i].$$

Remark

If the map is stratified with respect to the stratifications $(\mathfrak{X}, \mathfrak{Y})$ then we can choose Z_i to be strata from $\mathfrak{Y}$.