Part II: Recollement, or gluing

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The Topological Setting

- The setting: Y closed in X, $U := X - Y$.

$$Y \xrightarrow{i} X \xleftarrow{j} U \qquad \mathcal{D}_Y \xrightarrow{i_*} \mathcal{D}_X \xrightarrow{j^*} \mathcal{D}_U$$

- A t-structure on $\mathcal{D}_X$ determines ones on $\mathcal{D}_Y$ and $\mathcal{D}_U$:

$$\mathcal{D}_U^{\leq 0} = j^*(\mathcal{D}_X^{\leq 0}) \qquad \mathcal{D}_Y^{\leq 0} = (i^*)^{-1}(\mathcal{D}_X^{\leq 0})$$

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Basic Example

$$X = \mathbb{C} \qquad Y = \{0\} \qquad U = \mathbb{C}^\times$$

Example

The standard t-structures on $\mathcal{D}_U$ and $\mathcal{D}_Y$ yield the standard t-structure on $\mathcal{D}_X$.

Example

Glue the standard t-structure $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ to the shifted t-structure $(\mathcal{D}_Y^{\leq 1}, \mathcal{D}_Y^{\geq 1})$.

The General Setting: *Gluing Data*

$$\mathcal{D}_Y \xrightarrow{i_*} \mathcal{D}_X \xrightarrow{j^*} \mathcal{D}_U$$

1. i_* and j^* are exact and have left and right adjoints:

$$(i^*, i_* = i_!, i^!) \qquad (j_!, j^! = j^*, j_*)$$

2. $j^* i_* = 0$ but

$$i^* i_* = \mathrm{id}_{\mathcal{D}_Y} = i^! i_! \qquad j^* j_* = \mathrm{id}_{\mathcal{D}_U} = j^! j_!$$

3. For each $F \in \mathcal{D}_X$ there exist (unique) dist. triangles

$$j_! j^! F \to F \to i_* i^* F \xrightarrow{d} j_! j^! F[1]$$

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The Construction

Definition

Given t-structures on $\mathcal{D}_U, \mathcal{D}_Y$ and gluing data,

- $\mathcal{D}_X^{\leq 0} = \{K : j^*K \in \mathcal{D}_U^{\leq 0}, i^*K \in \mathcal{D}_Y^{\leq 0}\}$.
- $\mathcal{D}_X^{\geq 0} = \{K : j^!K \in \mathcal{D}_U^{\geq 0}, i^!K \in \mathcal{D}_Y^{\geq 0}\}$.

We say that the new t-structure on $\mathcal{D}_X$ is obtained by **gluing** or **recollement**.

It will be nondegenerate if those on $\mathcal{D}_U$ and $\mathcal{D}_Y$ are.

Example ($X = \mathbb{C}$, $Y = \{0\}$)

- F: constant sheaf $F = \mathbb{C}_X$ on X;

$$H(i^* F) = \mathbb{C}_Y \qquad H(i^! F) = \mathbb{C}_Y[-2]$$

- G: pushforward $G = j_* \mathbb{C}_U$ from U.

$$H(i^* G) = \mathbb{C}_Y \oplus \mathbb{C}_Y[-1] \qquad H(i^! G) = 0.$$

- Standard t-structures on U, Y, X:

F is in the heart $\mathcal{A}_X$; G is not.

- Recollement of $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ with $(\mathcal{D}_Y^{\leq 1}, \mathcal{D}_Y^{\geq 1})$:

F is *still* in the heart; now G is too.

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Functors Between Hearts

Gluing data (i_*, j^*) determine functors between hearts, e.g.

$$^p i_* = \tau_{\leq 0}\tau_{\geq 0} i_* \colon \mathcal{A}_Y \to \mathcal{A}_X.$$

- **Adjunctions**

$$({}^p i^*, {}^p i_* = {}^p i_!, {}^p i^!) \qquad ({}^p j_!, {}^p j^! = {}^p j^*, {}^p j_*).$$

- **Compositions** ${}^p j^* {}^p i_* = 0$ but

$${}^p i^* {}^p i_* = \mathrm{id}_{\mathcal{A}_Y} = {}^p i^! {}^p i_! \qquad {}^p j^* {}^p j_* = \mathrm{id}_{\mathcal{A}_U} = {}^p j^! {}^p j_!$$

- **Exact sequences**

$${}^p j_! {}^p j^! F \to F \to {}^p i_* {}^p i^* F \to 0$$

$$0 \to {}^p i_! {}^p i^! F \to F \to {}^p j_* {}^p j^* F.$$

Functors Between Hearts

${}^p i_*$ identifies $\mathcal{A}_Y$ with a full subcat of $\mathcal{A}_X$. For $F \in \mathcal{A}_X$:

- ${}^p i^* F$ is the largest quotient of F in $\mathcal{A}_Y$.
- ${}^p i^! F$ is the largest subobject of F in $\mathcal{A}_Y$.

Example ($X = \mathbb{C}$, $Y = \{0\}$)

Standard t-structures:

$$ {}^p i^* \mathbb{C}_X = \mathbb{C}_Y \qquad {}^p i^! \mathbb{C}_X = 0. $$

Recollement of $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ with $(\mathcal{D}_Y^{\leq 1}, \mathcal{D}_Y^{\geq 1})$:

$$ {}^p i^* \mathbb{C}_X = 0 \qquad {}^p i^! \mathbb{C}_X = 0. $$

Minimal Extensions

Definition

The *minimal extension* $j_{!*}F$ of $F \in \mathcal{A}_U$ is the unique $G \in \mathcal{A}_X$ such that

$$ {}^p j^* G = F \quad \text{and} \quad {}^p i^* G = {}^p i^! G = 0. $$

Example ($X = \mathbb{C}$, $Y = \{0\}$)

Standard t-structures: $j_{!*}\mathbb{C}_U = j_!\mathbb{C}_U$.
Recollement of $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ with $(\mathcal{D}_Y^{\leq 1}, \mathcal{D}_Y^{\geq 1})$:

$$ j_{!*}\mathbb{C}_U = \mathbb{C}_X. $$

Minimal Extensions — Construction

In fact

$$j_{!*}F = \text{image}[{}^pj_!F \to {}^pj_*F]$$

Here:

- For any $G \in \mathcal{A}_X$, adjunctions give

$$ {}^pj_!{}^pj^*G \to G \to {}^pj_*{}^pj^*G. $$

- If $G = {}^pj_!F$ this yields

$$ {}^pj_!F \longrightarrow {}^pj_*F. $$

The Theorem on Simple Objects

Theorem

Any simple object of $\mathcal{A}_X$ is of the form

$$^p i_* F \qquad or \qquad j_{!*} G$$

(with simple $F \in \mathcal{A}_Y$ or $G \in \mathcal{A}_U$).

Example ($X = \mathbb{C}$, $Y = \{0\}$)

Standard t-structures:

- sky-scraper sheaves $\mathbb{C}_x$.

Recollement of $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ with $(\mathcal{D}_Y^{\leq 1}, \mathcal{D}_Y^{\geq 1})$:

- sky-scrapers $\mathbb{C}_x$, $x \neq 0$, and $\mathbb{C}_0[-1]$.

Localisation

Given gluing data, one can make sense of the statements

- $\mathcal{D}_U = \mathcal{D}_X / \mathcal{D}_Y$.
- $\mathcal{A}_U = \mathcal{A}_X / \mathcal{A}_Y$.

[Gluing data $\iff$ $\mathcal{D}_Y \subset \mathcal{D}_U$ is a thick subcategory *and* the inclusion i_* has left and right adjoints.]

Summary

- A *t-structure* on a triangulated category $\mathcal{C}$ recovers:
  - an abelian category $\mathcal{A}$ (the *heart*)
  - a cohomological functor $\mathcal{C} \to \mathcal{A}$.
- *Gluing data* $\mathcal{D}_Y \to \mathcal{D}_X \to \mathcal{D}_U$ determine
  - a t-structure on $\mathcal{D}_X$ from those on $\mathcal{D}_Y, \mathcal{D}_U$
  - functors between the corresponding hearts, including
  - a notion of *minimal extension*.