ALTERNATIVE DEFN OF $D_G(X)$

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PROBLEM: EG is ∞-dim'l.

- Can't work in algebraic cats
- No Poncare Duality

NEW "STACKY" DEFN – Think about replacing X with approximations of $X \times EG$

DEF: • A RESOLUTION OF X is $\begin{array}{c} P \\ \downarrow \\ X \end{array}$ A G-EQUIV MAP

WHERE $G \curvearrowright P$ FREELY.

• $\mathrm{RES}_G(X)$ IS A CATEGORY.

IT HAS PRODUCTS

$$\begin{array}{ccc} & P \times_X Q & \\ \swarrow & & \searrow \\ Q & & P \\ \searrow & & \swarrow \\ & X & \end{array}$$

ALT. DEFN OF $D_G(X)$ CONT.

FOR $P \in Ob(RES_G(X))$, $\overline{P} \equiv P/G$

<u>EX:</u> • THE <u>TRIVIAL RESOLUTION</u> $T = G \times X \circlearrowleft G$ diagonal

$$\begin{array}{c} G \times X \\ \downarrow{\scriptstyle p_X} \\ X \end{array}$$

$\overline{T} \cong X$ canonically

• $G \circlearrowright M$ FREELY, then

$$\begin{array}{c} M \times X \\ \downarrow \\ X \end{array}$$

(LESS NAIVE)

<u>DEF:</u> $D^b_G(X, P)$ HAS OBJECTS

• $\mathcal{F}_X \in D^b(X)$

• $\overline{\mathcal{F}} \in D^b(\overline{P})$

• $\alpha : p^* \mathcal{F}_X \xrightarrow{\sim} q^* \overline{\mathcal{F}}$

$$\begin{array}{ccc} & P & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle q} \\ X & & \overline{P} \end{array}$$

MORPHISMS – COMPATIBLE MAPS

ALT DEFN OF $D_G(X)$ CONT

$D_G(X,P)$ IS NOT GOOD ENOUGH — BUT THE LIMIT OVER ALL P WORKS.

IF $P \xrightarrow{\varphi} Q$ AND $\mathcal{F} = (\mathcal{F}_X, \overline{\mathcal{F}}, \alpha) \in D^b_G(X,Q)$ THEN

$$\begin{tikzcd} & P \arrow[r, "\varphi"] \arrow[dl] \arrow[d] & Q \arrow[dll, bend left] \arrow[d] \\ X & \overline{P} \arrow[r, "\overline{\varphi}"] & \overline{Q} \end{tikzcd}$$

$$\varphi^* \mathcal{F} = (\mathcal{F}_X, \overline{\varphi}^* \overline{\mathcal{F}}, \varphi^*(\alpha)) \in D^b_G(X,P)$$

DEF: AN OBJECT OF $D^b_G(X)$ IS THE FOLLOWING DATA:

- FOR EACH $\begin{matrix} P \\ \downarrow \\ X \end{matrix}$ IN $Ob(\mathrm{Res}_G(X))$, A SHEAF $\mathcal{F}(P) \in D^b(\overline{P})$
- FOR EACH $Q \xrightarrow{\varphi} P$ IN $\mathrm{Res}_G(X)$, AN ISOM $\overline{\varphi}^* \mathcal{F}(P) \xrightarrow{\cong} \mathcal{F}(Q)$

SATISFYING NAT'L COMPATIBILITY REQUIREMENTS

IMPORTANT IDEAS

① $D^b_G(X,P) \neq D^b(\overline{P})$

Ex: $D^b_G(*,EG) = D^b(\mathrm{LocSys}(BG)) \neq D^b(\mathrm{Sh}(BG))$

HOW DO WE RECOVER $\mathcal{F}_X$? $\overline{T} \cong X$ SO $\mathcal{F}_X = \mathcal{F}(T)$ (CANONICALLY)

IF
$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & T \arrow[dl] \\ & X & \end{tikzcd}$$
THEN $\mathcal{F}(P) \in D^b_G(X,P)$. EVEN IF $\nexists\; P \to T$,

WE HAVE
$$\begin{tikzcd} & P \times_X T \arrow[dl] \arrow[dr] & \\ T \arrow[dr] & & P \arrow[dl] \\ & X & \end{tikzcd}$$
SO "IN THE LIMIT" WE LIVE IN $D^b_G(X,P)$

DEF: THE FORGETFUL FUNCTOR For: $D^b_G(X) \longrightarrow D^b(X)$

$$\mathcal{F} \longmapsto \mathcal{F}(T) \in D^b(\overline{T}) = D^b(X)$$

(AGREES W/ FORGETFUL $D_G(X,P) \longrightarrow D(X)$
$(\mathcal{F}_X, \mathcal{F}_\alpha) \longmapsto \mathcal{F}_X$)

IMPORTANT IDEAS

② WHEN IS $D^b_G(X,P)$ GOOD ENOUGH?

EX: IF X IS FREE,
$$\begin{array}{c} X \\ {\scriptstyle 1}\downarrow \\ X \end{array}$$
IS A FINAL OBJECT IN $\mathrm{Res}_G(X)$

SO $\mathcal{F}(P) = p^*\mathcal{F}(X)$

$\mathcal{F}(X) \in D^b(\overline{X})$.

$$D^b_G(X) \cong D^b(\overline{X})$$
$$\parallel\!\wr$$
$$D^b_G(X,X)$$

DEF: $P \longrightarrow X$ is <u>n-acyclic</u> $(n \in \mathbb{N})$ IF

- $\forall\ B \in Sh(X)$,
$$R^i f_*(f^*B) = \begin{cases} B & i=0 \\ 0 & i=1,2,\dots,n \\ ?? & i>n \end{cases}$$
- THIS IS TRUE AFTER BASE CHANGE $X' \longrightarrow X$.

IMPORTANT IDEAS

FACTS: (a)

$$\begin{tikzcd} & P \times_X Q \arrow[dl, "g"'] \arrow[dr] & \\ Q \arrow[dr] & & P \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

f n-acyclic $\Rightarrow$ g n-acyclic

(b) If $P \xrightarrow{\varphi} Q$ is n-acyclic map in $\mathrm{Res}_G(X)$

THEN $D_G^{[k,k+n]}(X,Q) \xrightarrow[\sim]{\varphi^*} D_G^{[k,k+n]}(X,P)$ $\forall k$.

(c) If $P \to X \leftarrow Q$ are both n-acyclic then

$D_G^{[k,k+n]}(X,P) \cong D_G^{[k,k+n]}(X,Q)$ canonically.

SO A SINGLE RESOLUTION P, n-acyclic, SUFFICES TO DEFINE $D_G^I(X)$ FOR AN n-INTERVAL I. ☆

LIMIT OF THESE IS $D_G^b(X)$

IMPORTANT IDEAS

③ COMBINING EARLIER REMARKS, WE OBSERVE THE FOLLOWING:

$\mathcal{F} \in D^b_G(X)$ IS DETERMINED BY ITS VALUES

- $\mathcal{F}(T)$ $\quad T$ TRIVIAL
- $\mathcal{F}(P_n)$, $n \in \mathbb{N}$ $\quad P_n$ SOME n-acyclic RESOLUTION.

IF NECESSARY, WE MAY USE CONVENIENT SUBCATS OF $RES_G(X)$ TO DEFINE $D^b_G(X)$ ☆

EX: IF EG EXISTS, $D^b_G(X) \cong D^b_G(X, EG)$ $\quad X \times EG$, $X \times G$ ARE ENOUGH

RMK: IF $\exists$ ∞-acyclic RESOLUTIONS, CAN TREAT $D^+_G(X)$ SIMILARLY.

FUNCTORS

WE WANT $f_*, f^*, f_!, f^!, \otimes, \mathcal{H}om, \mathbb{D}, \mathrm{Res}^G_H, \mathrm{Ind}^G_H, \dots$

SUBTLETY: $\mathcal{F} \in D_G(X)$ has data of $\alpha : \varphi^*\mathcal{F}(Q) \xrightarrow{\sim} \mathcal{F}(P)$

BUILT IN – NEED MORPHISM COMPATIBILITY, NOT JUST OBJECT COMPAT.

EX: DEFINE $\mathcal{F} \otimes \mathcal{G}$ BY • $(\mathcal{F} \otimes \mathcal{G})(P) = \mathcal{F}(P) \otimes \mathcal{G}(P) \in D^b(P)$

• FOR $P \xrightarrow{\varphi} Q$, WE NEED

$$\begin{tikzcd}
\varphi^*((\mathcal{F}\otimes\mathcal{G})(Q)) \arrow[r] \arrow[d, equal] & (\mathcal{F}\otimes\mathcal{G})(P) \arrow[d, equal] \\
\varphi^*(\mathcal{F}(Q)\otimes\mathcal{G}(Q)) \arrow[d, "\wr"', "\text{CAN}"] & \mathcal{F}(P)\otimes\mathcal{G}(P) \\
\varphi^*\mathcal{F}(Q)\otimes\varphi^*\mathcal{G}(Q) \arrow[ur] &
\end{tikzcd}$$

THE CANONICAL ISOM
$\varphi^*(A \otimes B) \cong \varphi^*A \otimes \varphi^*B$
ALLOWS US TO DEFINE THIS STRUCTURE ISOM, S.T. IT SATISFIES COMPAT. CONDITIONS

DEFINING THE FUNCTOR ON MORPHISMS REQUIRES SIMILAR COMPAT.

RMK: $\mathcal{H}om$ can be defined similarly

FUNCTORS

Suppose $X \xrightarrow{f} Y$. THIS INDUCES

$$f^\circ : \mathrm{RES}_G(Y) \longrightarrow \mathrm{RES}_G(X)$$

and moreover a map

$$\begin{pmatrix} P \\ \downarrow \\ Y \end{pmatrix} \longmapsto \begin{pmatrix} P \times_Y X \to P \\ \downarrow \quad\quad \downarrow \\ X \to Y \end{pmatrix} \qquad \overline{f^\circ P} \xrightarrow[\bar{f}_P]{} \overline{P}$$

Thus we can (namely) define $f_* : D_G(X) \longrightarrow D_G(Y)$ via

$$(f_* \mathcal{F})(P) = \bar{f}_{P*}(\mathcal{F}(f^\circ P)).$$

For any $P \xrightarrow{\varphi} Q$ we need a canonical isom

$$\begin{array}{ccc} \overline{f^\circ P} & \xrightarrow{\overline{f^\circ \varphi}} & \overline{f^\circ Q} \\ \bar{f}_P \downarrow & & \downarrow \bar{f}_Q \\ \overline{P} & \xrightarrow{\bar{\varphi}} & \overline{Q} \\ \swarrow & & \swarrow \\ X & \longrightarrow & Y \end{array} \qquad \bar{\varphi}^* \bar{f}_{Q*} \cong \bar{f}_{P*} \overline{f^\circ \varphi}^*.$$

$$\boxed{\text{BUT THIS DOES NOT HOLD IN GENERAL.}}$$

FUNCTORS | However, when φ is a smooth map, there is such an isom. SMOOTH BASE CHANGE.

FACT: When G is "nice", $\exists$ smooth n-acyclic resolutions $\forall n$.

Therefore $SRes_G(X)$ IS ENOUGH TO DEFINE $D_G(X)$

Objects: Smooth Resolutions Morphisms: All G-maps.

$\Rightarrow$ WE CAN DEFINE f_* NAIVELY FOR SMOOTH RESOLUTIONS

WHAT ABOUT f^*? $f^!$?

FACT: THE ESSENTIAL IMAGE OF $f^\circ(Res_G(Y))$ IN $Res_G(X)$ IS ENOUGH TO DEFINE $D_G(X)$

So $f^*\mathcal{F}(f^\circ P) \equiv f_\flat^*\mathcal{F}(P)$ WILL WORK, GIVEN SOME BASE CHANGE THEOREM.

SIMILAR DEFNS FOR $f_!, f^!$ $\Leftarrow$ HERE, $\dim P < \infty$ COMES IN HANDY.

FUNCTORS IDEA: RESTRICT TO NICE ENOUGH SUBCAT OF $\mathrm{Res}_G(X)$ AND NAIVE DEFNS WORK.

DEF: $H \subset G$. Then $\mathrm{Res}_H(X) \xrightarrow{\mathrm{ind}} \mathrm{Res}_G(X)$
$G \curvearrowright X$

$$P \longmapsto G \times_H P$$

MOREOVER $\overline{G \times_H P} = G \backslash G \times_H P \underset{\theta}{\overset{\sim}{\longleftrightarrow}} H \backslash P = \overline{P}$ CANONICALLY.

SO DEFINE $R^G_H : D_G(X) \longrightarrow D_H(X)$

$$R^G_H \mathcal{F}(P) = \theta^* \mathcal{F}(\mathrm{ind}(P))$$

AN ISOM, USUALLY IGNORED

Ex: $H = \{e\}$. $R^G_H \mathcal{F}$ IS DETERMINED ON X, AND $R^G_H \mathcal{F}(X) = \mathcal{F}(G \times X) \overset{\text{NOT QUITE}}{=} \mathcal{F}(T)$
$= \mathrm{For}\,\mathcal{F}$.

FUNCTORS

INDUCTION EQUIVALENCE: $H \subset G$, $H \circlearrowright X$ THEN $\mathrm{Res}_H(X) \xrightarrow{\sim} \mathrm{Res}_G(G \times_H X)$

$$P \longmapsto G \times_H P$$

$$\bar{P} \xrightarrow{\sim} \overline{G \times_H P}$$

SO $D_H(X) \cong D_G(G \times_H X)$

Ex: $D_H(*) \cong D_G(G/H)$

INDUCTION FUNCTORS: $H \subset G$, $G \circlearrowright X$, $G \times_H X \xrightarrow{a} X$. THEN

$I^G_{H*} : D_H(X) \longrightarrow D_G(X)$ IS $D_H(X) \xrightarrow{\sim} D_G(G \times_H X) \xrightarrow{a_*} D_G(X)$

$I^G_{H!}$ " — " $a_!$

PROP: $I^G_{H!} \vdash R^G_H \vdash I^G_{H*}$

FUNCTORS

QUOTIENT EQUIVALENCE

$H \trianglelefteq G$
$G \curvearrowright X$
$H \curvearrowright X$ freely

$X \to {}_H\backslash X$
$G/H \circlearrowright {}_H\backslash X$

$$\mathrm{Res}_G(X) \xrightarrow{\sim} \mathrm{Res}_{G/H}({}_H\backslash X)$$
$$P \longmapsto {}_H\backslash P$$

AND

$$\overline{P} \xrightarrow{\sim} \overline{{}_H\backslash P}$$

SO

$$D_G(X) \cong D_{G/H}({}_H\backslash X)$$

SO WHAT DOESN'T WORK NAIVELY?

NON-EQUIV: $C_X \equiv p_X^* C_*$ $\quad \mathcal{D}_X \equiv p_X^! C_*$ WHERE $X \xrightarrow{p_X} *$

EQUIV: $C_{X,G} \equiv p_X^* C_{*,G}$ $\quad C_{*,G}(P) = C_{\bar{P}}$ SO $C_{X,G}(Q) = C_{\bar{Q}}$

$\mathcal{D}_{X,G} \equiv p_X^! C_{*,G}$. BUT $\mathcal{D}_{*,G} = C_{*,G}$ SO $\mathcal{D}_{*,G}(P) = C_{\bar{P}} \neq \mathcal{D}_{\bar{P}}$ FOR A SMOOTH FREE G-SPACE P.

WHAT IS $\mathcal{D}_{X,G}$? AS WE DEFINED IT EARLIER ON $p_X^\circ \mathrm{Res}_G(*) = \{X \times P$

$$\begin{tikzcd} X\times P \arrow[r] \arrow[d] & P \arrow[d] \\ X \arrow[r] & * \end{tikzcd} \qquad \overline{X\times P} \xrightarrow{f} \bar{P}$$

CLAIM: $\mathcal{D}_{X,G}(X\times P) = f^! C_{\bar{P}} \simeq \mathcal{D}_{\overline{X\times P}} \otimes (f^*\mathcal{D}_{\bar{P}})^\vee$

Pf: EXERCISE, USE TOPOLOGICALLY FIBERED STUFF FROM DUALITY TALK, PROPERTIES OF INVERTIBLE SHEAVES.

RMK: $(f^*\mathcal{D}_{\bar{P}})^\vee$ IS A SHEAF IN POSITIVE HOMOLOGICAL DEGREE!

RMK: $\mathrm{For}(\mathcal{D}_{X,G}) = \mathcal{D}_{X,G}(T) = \mathcal{D}_X$ SINCE $\underset{\overline{X\times G}}{\underset{\parallel}{\bar{T}}} = X \xrightarrow{f} \underset{\bar{G}}{\underset{\parallel}{\bar{P}}} = *$

DUALITY

DEF: $\mathbb{D} \equiv \mathcal{H}om(\cdot, \mathcal{D}_{X,G})$

THM:

SET I	SET II	SET III
$f_!, f_*, f^!, f^*, \otimes, \mathcal{H}om$	R^G_H, Ind Equiv, Quot Equiv (NOT I^G_{H*}, $I^G_{H!}$)	$\mathbb{D}$

- Functors in $\boxed{I}$ have usual props. (Naive maps on P)
- Functors in $\boxed{I}$ commute w/ functors in $\boxed{II}$ (Check on $Res_X(G)$)
- $\mathbb{D}^2 \cong Id$, $\quad \mathbb{D} f_* = f_! \mathbb{D}$, $\quad \mathbb{D} \circ For = For \circ \mathbb{D}$
- If $X \xrightarrow{\nu} Y$ smooth G-Map, $\cancel{\mathbb{D}_\nu \equiv \nu^! \mathcal{D}_{Y,G}}$ $\mathcal{D}_\nu \equiv \nu^! C_{Y,G}$

  THEN $\mathbb{D} \nu^* = \nu^* \underbrace{\mathbb{D}}_{=\nu^! \mathbb{D}} \otimes \mathcal{D}_\nu$

Ex: By calculating $\mathcal{D}_\nu$ (easily) one sees:

① Quotient equivalence when G conn

For $q^*: D^b_{G/H}({}_H X) \xrightarrow{\sim} D^b_G(X)$

$\mathbb{D} q^* = q^* \mathbb{D}[\dim H]$

② Induction equiv, H conn. $\quad D^b_G(G \times_H X) \xrightarrow{\theta} D^b_H(X)$

$\mathbb{D}\theta = \theta \mathbb{D}[\dim H - \dim G]$

SMOOTH BASE CHANGE

BONUS SLIDE$_1$

THM: ① SMOOTH BASE CHANGE COMMUTES W/ $\otimes, \mathcal{H}om, f_*, f_!, f^*, f^!$

EXPLICITLY - $T \xrightarrow{\nu} S$ smooth. ν^* will denote $D^b(S) \xrightarrow{\nu^*} D^b(T)$

OR

$D^b(X) \longrightarrow D^b(X \times_S T)$

for any $X \to S$.

THEN $\forall\ A, B \in D^b(Y)$, $C \in D^b(X)$, $\begin{matrix} Y \\ \downarrow \\ S \end{matrix}$, $\begin{matrix} X & \xrightarrow{f} & Y \\ & \searrow \ \swarrow & \\ & S & \end{matrix}$ WE HAVE

$\nu^*(A \otimes B) \cong \nu^* A \otimes \nu^* B$

$\nu^* \mathcal{H}om(A, B) \cong \mathcal{H}om(\nu^* A, \nu^* B)$

$\nu^* f_*(C) \cong f_* \nu^*(C)$

$\nu^* f^*(A) \cong f^* \nu^*(A)$

SAME W/ $f_!, f^!$

② **RMK:** ν^* CAN <u>NOT</u> COMMUTE WITH $\mathbb{D}$ SINCE $\nu^* \mathcal{D}_X \not\cong \mathcal{D}_{X \times_S T}$

THE DUALIZING SHEAF

$\nu^* \mathbb{D}(A) = \nu^* \mathcal{H}om(A, \mathcal{D}_X) = \mathcal{H}om(\nu^* A, \nu^* \mathcal{D}_X)$

SMOOTH BASE CHANGE

BONUS SLIDE 2

FACT: FOR A SMOOTH MAP $\nu : X \to Y$, $\mathcal{F} \in D^b(Y)$ WE HAVE

$$\nu^! \mathcal{F} \cong \nu^* \mathcal{F} \otimes D_\nu \qquad \text{WHERE} \qquad D_\nu \equiv \nu^! \mathbb{C}_Y$$

D_ν IS <u>INVERTIBLE</u> (i.e. IT IS LOCALLY ISOM TO $\mathbb{C}_X[\dim X - \dim$

SO IT HAS A DUAL SHEAF $D_\nu^\vee$ AND

$$\mathcal{H}om(A, B \otimes D_\nu) \cong \mathcal{H}om(A, B) \otimes D_\nu \cong \mathcal{H}om(A \otimes D_\nu^\vee, B)$$

THEREFORE

$$\mathbb{D}(\nu^* A) \cong D_\nu \otimes \nu^* \mathbb{D}(A)$$

CALCULATING D_ν (EQUIVARIANTLY) WILL GIVE US MORE EXPLICIT IDENTITIES

SEE SLIDE 15