

ALTERNATIVE DEFIN OF $D_G(X)$

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PROBLEM: EG is ∞ -dim'l.

- Can't work in algebraic cats
- No Poincare Duality

NEW "STACKY" DEFIN - Think about replacing X with approximations of $X \times EG$

DEF: • A RESOLUTION of X is

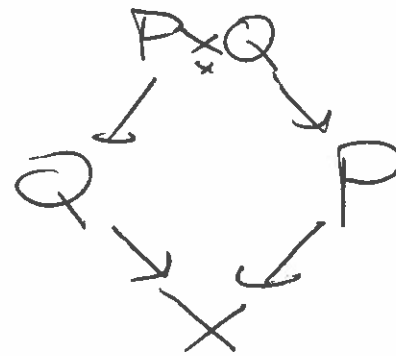


A G -EQUIV MAP

WHERE $G \curvearrowright P$ FREELY.

• $RES_G(X)$ IS A CATEGORY.

IT HAS PRODUCTS



ALT. DEFN OF $D_G(X)$ CONT.

For $P \in \text{Ob}(\text{RES}_G(X))$, $\bar{P} \equiv P/G$

Ex: • THE TRIVIAL RESOLUTION

$$T = G \times X \supset G \text{ diagonal}$$



$\bar{T} \cong X$ canonically

• GCM FREELY, then



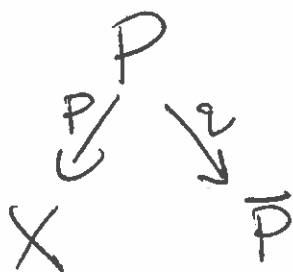
LESS
(NAIVE)

DEF: $D_G^b(X, P)$ HAS OBJECTS

• $\mathcal{F}_X \in D^b(X)$

• $\bar{\mathcal{F}} \in D^b(\bar{P})$

• $\alpha: p^* \mathcal{F}_X \xrightarrow{\sim} q^* \bar{\mathcal{F}}$



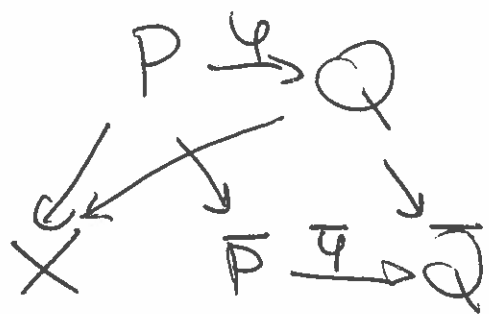
MORPHISMS

- COMPATIBLE MAPS

ALT DEFN OF $D_G^b(X)$ CONT

$D_G^b(X, P)$ IS NOT GOOD ENOUGH — BUT THE LIMIT OVER ALL P WORKS.

IF $P \xrightarrow{\psi} Q$ AND $\mathcal{F} = (\mathcal{F}_X, \overline{\mathcal{F}}, \alpha) \in D_G^b(X, Q)$ THEN



$$\psi^* \mathcal{F} = (\mathcal{F}_X, \overline{\psi^* \mathcal{F}}, \psi^* \alpha) \in D_G^b(X, P)$$

DEF: AN OBJECT OF $D_G^b(X)$ IS THE FOLLOWING DATA:

- FOR EACH $\begin{array}{c} P \\ \downarrow \\ X \end{array} \text{ IN } \text{IND}(\text{RES}_G(X))$ A SHEAF $\mathcal{F}(P) \in D^b(\overline{P})$
- FOR EACH $Q \xrightarrow{\psi} P$ IN $\text{RES}_G(X)$, AN ISOM $\psi^* \mathcal{F}(P) \xrightarrow{\cong} \mathcal{F}(Q)$
SATISFYING NAT'L COMPATIBILITY REQUIREMENTS

IMPORTANT IDEAS

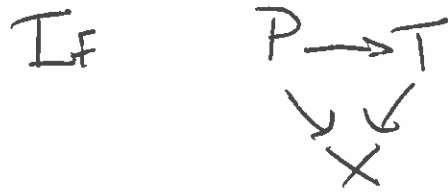
① $D_G^b(X, P) \neq D^b(\bar{P})$

Ex: $D_G^b(*, EG) = D^b(\text{LocSps}(BG))$

$D^b(\text{Sh}(BG))$

How do we recover \mathcal{F}_x ?

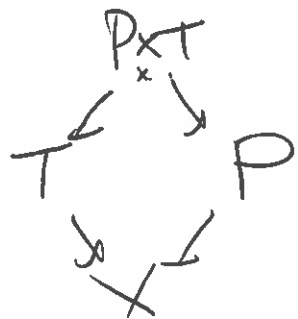
$\bar{T} \cong X$ so $\mathcal{F}_x = \mathcal{F}(T)$
(CANONICALLY)



THEN $\mathcal{F}(P) \in D_G^b(X, P)$

EVEN IF $\nexists P \rightarrow T$

WE HAVE



SO "IN THE LIMIT" WE LIVE IN $D_G^b(X, \mathcal{F}$

DEF: THE FORGETFUL FUNCTOR

For: $D_G^b(X) \rightarrow D^b(X)$

$\mathcal{F} \mapsto \mathcal{F}(T) \in D^b(\bar{T}) = D^b(X)$

(AGREES W/ FORGETFUL

$D_G^b(X, P) \rightarrow D(X)$
 $(\mathcal{F}_x, \mathcal{F}_x) \rightarrow \mathcal{F}_x$)

IMPORTANT IDEAS

② WHEN IS $D_G^b(X, P)$ GOOD ENOUGH?

EX: IF X IS FREG, $\begin{matrix} X \\ \cong \downarrow \\ X \end{matrix}$ IS A FINAL OBJECT IN $\text{RES}_G(X)$

SO $F(P) = P^*F(X)$
 $F(X) \in D^b(\bar{X})$.

$D_G^b(X) \cong D^b(\bar{X})$
 $\neq D_G^b(X, X)$

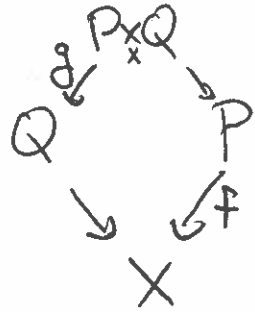
DEF: $P \rightarrow X$ IS n-acyclic ($n \in \mathbb{N}$) IF

- $\forall B \in \text{Sh}(X)$, $R^i f_* (f^* B) = \begin{cases} B & i=0 \\ 0 & i=1, 2, \dots, n \\ ?? & i > n \end{cases}$
- THIS IS TRUE AFTER BASE CHANGE $X' \rightarrow X$.

IMPORTANT IDEAS

FACTS:

(a)



f n -acyclic \Rightarrow g n -acyclic

(b) IF $P \xrightarrow{\psi} Q$ IS n -ACYCLIC MAP IN $\text{RES}_G(X)$

THEN $D_G^{[k, k+n]}(Q) \xrightarrow[\cong]{\psi^*} D_G^{[k, k+n]}(X, P) \quad \forall k.$

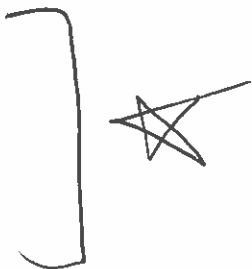
(c) IF $P \rightarrow X \leftarrow Q$ ARE BOTH n -ACYCLIC THEN

$D_G^{[k, k+n]}(X, P) \cong D_G^{[k, k+n]}(X, Q)$ canonically.

SO A SINGLE RESOLUTION P , n -ACYCLIC, SUFFICES TO

DEFINE $D_G^I(X)$ FOR AN n -INTERVAL I .

LIMIT OF THESE IS $D_G^b(X)$



IMPORTANT IDEAS

③ COMBINING EARLIER REMARKS, WE OBSERVE THE FOLLOWING:

$F \in D_G^b(X)$ IS DETERMINED BY ITS VALUES

- $F(T)$ T TRIVAL
- $F(P_n)$, $n \in \mathbb{N}$ P_n SOME n -acyclic RESOLUTION.

IF NECESSARY, WE MAY USE CONVENIENT SUBCATS
OF $\text{RES}_G(X)$ TO DEFINE $D_G^b(X)$ ★

EX: IF EG EXISTS, $D_G^b(X) \cong D_G^b(X, EG)$ $X \times EG$, $X \times G$ ARE ENOUGH

RMK: IF \exists ∞ -acyclic RESOLUTIONS, CAN TREAT $D_G^+(X)$ SIMILARLY.

FUNCTORS

WE WANT f_* , f^* , $f_!$, $f^!$, \otimes , Hom , \mathbb{D} , Res_H^G , Ind_H^G, \dots

SUBTLETY: $F \in \mathcal{D}_G(X)$ has data of $\alpha: \varphi^* F(Q) \xrightarrow{\sim} F(P)$

BUILT IN - NEED MORPHISM COMPATIBILITY, NOT JUST OBJECT COMPAT.

EX: DEFINE $F \otimes G$ BY $\cdot (F \otimes G)(P) = F(P) \otimes G(P) \in \mathcal{D}^b(P)$

• FOR $P \xrightarrow{\varphi} Q$, WE NEED $\varphi^*(F \otimes G(Q)) \rightarrow (F \otimes G)(P)$

\parallel

$\varphi^*(F(Q) \otimes G(Q)) \rightarrow F(P) \otimes G(P)$

\parallel

$\varphi^* F(Q) \otimes \varphi^* G(Q)$

|| CAN

THE CANONICAL ISOM
 $\varphi^*(A \otimes B) \cong \varphi^* A \otimes \varphi^* B$
ALLOWS US TO DEFINE THIS
STRUCTURE ISOM, ST.
IT SATISFIES COMPAT. CONDITIONS

DEFINING THE FUNCTOR ON
MORPHISMS REQUIRES SIMILAR
COMPAT.

RMK: Hom can be defined similarly

FACTORS

Suppose $X \xrightarrow{f} Y$.

THIS INDUCES

$$f^\circ: \text{RES}_G(Y) \rightarrow \text{RES}_G(X)$$

and Moreover a map

$$\begin{pmatrix} P \\ \downarrow \\ Y \end{pmatrix} \mapsto \begin{pmatrix} P \times X & \rightarrow & P \\ \downarrow & & \downarrow \\ X & \rightarrow & Y \end{pmatrix}$$

$$\overline{f^\circ P} \xrightarrow{\overline{f}_P} \overline{P}$$

Thus we can ^(nearly) define

$$f_*: D_G(X) \rightarrow D_G(Y) \quad \text{via}$$

$$(f_* F)(P) = \overline{f}_*(F(f^\circ P)).$$

For any $P \xrightarrow{\psi} Q$ we need a canonical isom

$$\begin{array}{ccc} \overline{f^\circ P} & \xrightarrow{\overline{f^\circ \psi}} & \overline{f^\circ Q} \\ \overline{f}_P \downarrow & & \downarrow \overline{f}_Q \\ \overline{P} & \xrightarrow{\overline{\psi}} & \overline{Q} \end{array}$$

$$\psi^* \overline{f}_{Q*} \cong \overline{f}_{P*} \overline{f}^{\circ \psi*}$$

BUT THIS DOES NOT HOLD IN GENERAL.

FUNCTORS

However, when φ is a smooth map, there is such an isom.

SMOOTH BASE CHANGE.

FACT: When G is "nice", \exists smooth n -acyclic resolutions $\forall n$.

Therefore $\text{SR}_{\text{RES}_G(X)}$ IS ENOUGH TO DEFINE $D_G(X)$

\Rightarrow WE CAN DEFINE f_* NAIVELY FOR SMOOTH RESOLUTIONS

\uparrow Objects: Smooth Resolutions Morphisms: All Groups.

WHAT ABOUT f^* ? $f^!$?

FACT: THE ESSENTIAL IMAGE OF $f^0(\text{RES}_G(Y))$ IN $\text{RES}_G(X)$ IS ENOUGH TO DEFINE $D_G(X)$

So $f^* \mathcal{F}(f^0 P) \cong f^* \mathcal{F}(P)$ WILL WORK, GIVEN SOME BASE CHANGE THEOREM.

SIMILAR DEFINITIONS FOR $f_!, f^!$ \leftarrow HERE, $\dim P < \infty$ COMES IN HANDY.

FUNCTIONS | IDEA: RESTRICT TO NICE ENOUGH SUBCAT OF $\text{RES}_G(X)$ AND NAIVE DEFINES WORK.

DEF: $H \subset G$. Then $\text{RES}_H(X) \xrightarrow{\text{ind}} \text{RES}_G(X)$
 $G \curvearrowright X \quad P \mapsto G \times_H P$

MOREOVER

$$\overline{G \times_H P} = G / G \times_H P \xrightarrow[\theta]{\sim} H / P = \mathbb{F} \quad \text{CANONICALLY.}$$

SO DEFINING $R_H^G: D_G(X) \rightarrow D_H(X)$

$$R_H^G \mathcal{F}(P) = \theta^* \mathcal{F}(\text{ind}(P))$$

\uparrow AN ISOM, USUALLY IGNORED

EX: $H = \{e\}$. $R_H^G \mathcal{F}$ IS DETERMINED ON X , AND $R_H^G \mathcal{F}(X) = \mathcal{F}(G \times X) \stackrel{\text{NOT QUITE}}{\downarrow} = \mathcal{F}(T) = \text{For } \mathcal{F}$.

FUNCTORS

INDUCTION EQUIVALENCE

$$H \subset G \\ H \subset X$$

$$\text{SO } D_H(X) \cong D_G(G \times_H X)$$

$$\underline{\text{Ex:}} \quad D_H(*) \cong D_G(G/H)$$

$$\text{THEN } \text{RES}_H(X) \xrightarrow{\sim} \text{RES}_G(G \times_H X)$$

$$P \longmapsto G \times_H P$$

$$P \xrightarrow{\sim} \overline{G \times_H P}$$

INDUCTION FUNCTORS

$$H \subset G \\ G \subset X$$

$$I_{H*}^G : D_H(X) \longrightarrow D_G(X)$$

$$I_H^G$$

$$G \times_H X \xrightarrow{a} X \quad \text{THEN}$$

$$\text{IS } D_H(X) \xrightarrow{\sim} D_G(G \times_H X) \xrightarrow{a_*} D_G(X)$$

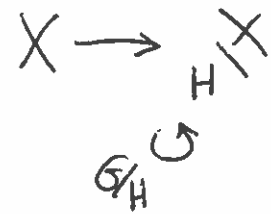
$a_!$

$$\underline{\text{PROP:}} \quad I_H^G \vdash R_H^G \vdash I_{H*}^G$$

FUNCTORS

QUOTIENT EQUIVALENCE

$H \trianglelefteq G$
 $G \curvearrowright X$
 $H \curvearrowright X$ freely



$$\begin{array}{ccc}
 \text{RES}_G(X) & \xrightarrow{\sim} & \text{RES}_{G/H}(H \backslash X) \\
 P & \longmapsto & H/P
 \end{array}$$

AND

$$\mathbb{P} \xrightarrow{\sim} \#/\mathbb{P}$$

SO

$$D_G(X) \cong D_{G/H}(H \backslash X)$$

DUALITY

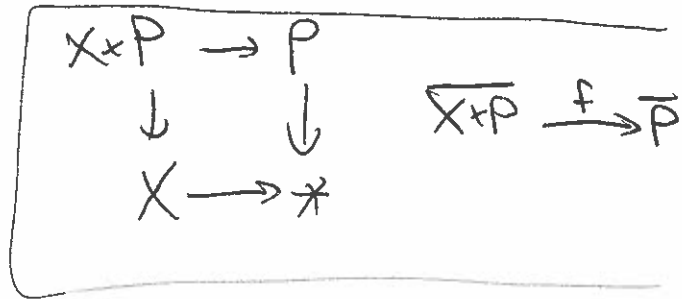
SO WHAT DOESN'T WORK NAIVELY?

NON-EQUIV: $C_X \equiv P_X^* C_*$ $\mathcal{D}_X \equiv P_X^! C_*$ WHERE $X \xrightarrow{P} *$

EQUIV: $C_{X,G} \equiv P_X^* C_{*,G}$ $C_{*,G}(P) = C_P$ SO $C_{X,G}(Q) = C_Q$

$\mathcal{D}_{X,G} \equiv P_X^! C_{*,G}$ BUT $\mathcal{D}_{*,G} = C_{*,G}$ SO $\mathcal{D}_{X,G}(P) = C_P \neq \mathcal{D}_P$ FOR A SMOOTH FREE G-SPACE P.

WHAT IS $\mathcal{D}_{X,G}$? AS WE DEFINED IT EARLIER ON $P_X^! \text{RES}_G(*) = \{X \times P\}$



CLAIM: $\mathcal{D}_{X,G}(X \times P) = f^! C_P = \mathcal{D}_{X \times P} \otimes (f^* \mathcal{D}_P)^\vee$

PF: EXERCISE, USE TOPOLOGICALLY FIBERED STUFF FROM DUALITY TALK, PROPERTIES OF INVERTIBLE SHEAVES.

RMK: $(f^* \mathcal{D}_P)^\vee$ IS A SHEAF IN POSITIVE HOMOLOGICAL DEGREE!

RMK: For $(\mathcal{D}_{X,G}) = \mathcal{D}_{X,G}(T) = \mathcal{D}_X$ SINCE $\begin{matrix} \overline{T} = X & \xrightarrow{f} & \overline{P} = * \\ \parallel & & \parallel \\ X \times G & & G \end{matrix}$

DUALITY

DEF: $\mathbb{D} \equiv \text{Hom}(\cdot, \mathcal{D}_{X,G})$

THM: SET I

$f_!, f_*, f^!, f^*, \otimes, \text{Hom}$

SET II

R_H^G , Ind Equiv, Quot Equiv

(NOT $I_{H \times 0}^G, I_{H!}^G$)

SET III

\mathbb{D}

• FUNCTORS IN $\boxed{\text{I}}$ HAVE USUAL PROPS. (NAIVE MAPS ON \overline{P})

• FUNCTORS IN $\boxed{\text{I}}$ COMMUTE W/ FUNCTORS IN $\boxed{\text{II}}$ (CHECK ON $\text{RES}_X(G)$)

• $\mathbb{D}^2 \cong \text{Id}$, $\mathbb{D}f_* = f_! \mathbb{D}$, $\mathbb{D} \circ f_{or} = f_{or} \circ \mathbb{D}$

• IF $X \xrightarrow{\nu} Y$ SMOOTH G -MAP, ~~$\mathbb{D}_\nu \equiv \nu^! \mathbb{D}_Y$~~ $\mathbb{D}_\nu \equiv \nu^! C_{Y,G}$

THEN

$\mathbb{D} \nu^* = \nu^* \mathbb{D} \otimes \mathbb{D}_\nu$
 $\nu^! \mathbb{D}$ $\xrightarrow{\nu^! \mathbb{D}}$

① QUOTIENT EQUIVALENCE WHEN G CAN

FOR $q^*: D_{G/H}^b(H^X) \xrightarrow{\sim} D_G^b(X)$

$\mathbb{D} q^* = q^* \mathbb{D} [\dim H]$

EX: BY CALCULATING \mathbb{D}_ν (EASILY) ONE SEES:

② INDUCTION EQUIV, H CAN. $D_G^b(G \times X) \xrightarrow{\theta} D_H^b(X)$

$\mathbb{D} \theta = \theta \mathbb{D} [\dim H - \dim G]$

SMOOTH BASE CHANGE

BONUS SLIDE

THM: ① SMOOTH BASE CHANGE COMMUTES w/ $\otimes, \text{Hom}, f_*, f_!, f^*, f^!$

EXPLICITLY - $T \xrightarrow{v} S$ smooth. v^* will denote $D^b(S) \xrightarrow{v^*} D^b(T)$ or $D^b(X) \rightarrow D^b(X \times_S T)$ for any $X \rightarrow S$.

THEN $\forall A, B \in D^b(Y), C \in D^b(X)$, $Y \downarrow S$, $X \xrightarrow{f} Y$, $X \downarrow S$

WE HAVE

$$v^*(A \otimes B) \cong v^*A \otimes v^*B$$

$$v^*f_*(A) \cong f_*v^*(A)$$

$$v^*\text{Hom}(A, B) \cong \text{Hom}(v^*A, v^*B)$$

$$v^*f^*(A) \cong f^*v^*(A)$$

SAME w/ $f_!, f^!$

②

RMK: v^* CAN NOT COMMUTE WITH ID SINCE $v^*D_X \not\cong D_{X \times_S T}$

$$v^*ID(A) = v^*\text{Hom}(A, D_X) = \text{Hom}(v^*A, v^*D_X)$$

THIS DUALIZING SHEAF

SMOOTH BASE CHANGES

BONUS SLIDES

FACT: For a smooth map $\nu: X \rightarrow Y$, $F \in D^b(Y)$ we have

$$\nu^! F \cong \nu^* F \otimes D_\nu \quad \text{WHERE } D_\nu \equiv \nu^! \mathcal{O}_Y$$

D_ν IS INVERTIBLE (i.e. IT IS LOCALLY ISOM TO $\mathcal{O}_X[\dim X - \dim Y]$)

SO IT HAS A DUAL SHEAF D_ν^\vee AND

$$\text{Hom}(A, B \otimes D_\nu) \cong \text{Hom}(A, B) \otimes D_\nu \cong \text{Hom}(A \otimes D_\nu^\vee, B)$$

THEREFORE

$$\mathbb{D}(\nu^* A) \cong D_\nu \otimes \nu^* \mathbb{D}(A)$$

CALCULATING D_ν (EQUIVARIANTLY) WILL GIVE US MORE EXPLICIT IDENTITIES

~~SHOW~~

SEE SLIDE (15)