Introduction to the equivariant world

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Isle of Skye

May 26, 2010.

Universal principal bundles

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- Theorem: EG exists for all topological group G, and unique up to equivariant homotopy.

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- EG is contractible. In fact, if P is a contractible space with a free G-action then $P \to P/G$ is a universal principle G-bundle.
- Theorem: EG exists for all topological group G, and unique up to equivariant homotopy.
- Example: $\mathbb{C}^\infty \to \mathbb{P}^\infty(\mathbb{C})$ is a universal principle $\mathbb{C}^*$ bundle. Similarly,

$$BGL_n = Hom(\mathbb{C}^n, \mathbb{C}^\infty)/GL_n = Gr(n, \infty), BB_n = BT_n = Flag(1, 2, \dots, n, \infty)$$

.

- $EGL_n \times_{GL_n} \mathbb{C}^n \to BGL_n$ is a universal vector bundle, any vector bundle can be pulled back from this.

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Properties

1. $f : X \to Y$ G-map induces $H(f) : H_G(Y) \to H_G(X)$
2. $h : G \to H$ homomorphism, then EH can serve as EG and we have a projection $EH \times_G X \to EH \times_H X$ which induces $H(h) : H_H(X) \to H_G(X)$
3. $H_G^*(pt) = H^*(BG) = \mathbb{C}[\mathfrak{h}]^W$, and $H_{\mathfrak{g}}^*(X)$ is a $H_G^*(pt)$-module. For example $H_{GL_n}^*(pt) = S^W = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$.

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$$(d_G\alpha)(X) = (d - \iota(X_M))\alpha(X)$$

which increases the degree by one if the $\mathbb{Z}$-grading is given by

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Note that $\alpha \in \Omega_G(M)$ is equivariantly closed if

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In other words:

$$\int_M \alpha(X) = (2\pi)^l \sum_{p \in M^T} \frac{\alpha(X)_0(p)}{\prod_i \lambda_i}$$

where λ_i are the weights of the Lie action

$$X : \xi \in T_p M \to [X_M(p), \xi] \in T_p M.$$

How many lines intersect 2 given lines and go through a point in $\mathbb{C}^3$?

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- The weights on $T_{(i,j)}Gr$ are $\mu_s - \mu_i, \mu_s - \mu_j$ with $s \neq i,j$. ABBV localization gives

$$\int_{Gr(2,4)} c_1^2 c_2 = \sum_{\sigma \in S_4/S_2} \sigma \cdot \frac{(\mu_1+\mu_2)^2 \mu_1 \mu_2}{(\mu_3-\mu_1)(\mu_4-\mu_1)(\mu_3-\mu_2)(\mu_4-\mu_2)} = 2$$

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Then
$f(g_1(g_2(x))) = f(x) = f((g_1 g_2)x) \Leftrightarrow (m \times id_X)^*(p^* f) = (id_G \times \pi)^*(\pi^* f)$

Here come equivariant sheaves

Hint: "G-equivariant sheaf $\mathcal{F}$ on $X=$ sheaf whose sections over an open set are G-invariant"

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Definition

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a G-variety X (or a sheaf of vector spaces on a topological G-space X) is G-equivariant if

1. *There is a given isomorphism of sheaves on $G \times X$ $I : \pi^*\mathcal{F} \simeq m^*\mathcal{F}$*
2. *$(m \times id_X)^* I = p_{23}^* I \circ (id_G \times \pi)^* I$ where $p_{23} : G \times G \times X \to G \times X$ is a the projection along the first factor.*

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2. *$(m \times id_X)^* I = p_{23}^* I \circ (id_G \times \pi)^* I$ where $p_{23} : G \times G \times X \to G \times X$ is a the projection along the first factor.*

Remark

- *The equivariant structure is given by $(\mathcal{F}, I)$, so asking if a sheaf $\mathcal{F}$ is equivariant is meaningless. I is not necessarily unique.*
- *does not follows from as for invariant functions.*
- *If $\mathcal{F}$ is locally free, i.e $\mathcal{F}$ is a vector bundle, then the equivariant structure is equivalent to a linear fiberwise action of G on $\mathcal{F}$.*

Hint: "G-equivariant sheaf $\mathcal{F}$ on $X=$ sheaf whose sections over an open set are G-invariant"

Definition

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- *If G acts (topologically) freely on X (i.e an open neighbourhood looks like $G \times U$ acted on by G on the first factor) then*

$$G\text{-equivariant sheaves on } X = \text{ sheaves on } X/G$$

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Definition

Bernstein and Lunts-topological version Consider the diagram

$$X \xleftarrow{p} X \times EG \xrightarrow{q} X_G = EG \times_G X$$

$D^{+,b}_G(X)$ is the full subcategory of $D^{+,b}(EG \times_G X)$ consisting of complexes $\mathcal{F} \in D^{+,b}(EG \times_G X)$ such that $q^*\mathcal{F} \simeq p^*\mathcal{G}$ for some $\mathcal{G} \in D^{+,b}(X)$.

Remark

1. *In other words, an equivariant sheaf is a triple* $(\mathcal{G}, \mathcal{F}, \alpha)$ *where*

$$\mathcal{G} \in D^{b,+}(X), \mathcal{F} \in D^b(EG \times_G X), \alpha : \; p^*\mathcal{G} \xrightarrow{\simeq} q^*\mathcal{F} \; .$$

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2. *This definition does not work in the algebraic category because* EG *is infinite dimensional. Technical problem: There is no* G*-variety* P *with free action such that* $P \to X$ *is* ∞*-acyclic, i.e the fibers have trivial cohomology. Solution: Approximation of* EG *with finite dimensional varieties* EG_n *where* $EG_n \times_G X \to X$ *is* n*-acyclic.*