Introduction to Verdier Duality

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Introduction

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- The exact functor $f^* : Sh_k(Y) \to Sh_k(X)$

where $f_!$ is the *direct image with compact support*

$$f_!\mathcal{F}(U) := \{s \in \mathcal{F}(f^*(U)) | f : supp(s) \hookrightarrow Y \text{ is proper}\}$$

a subsheaf of $f_*\mathcal{F}$.

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In general, such a functor does not exist at the level of sheaves.

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(Global) Verdier Duality

There exists a right adjoint $f^! : D^+(Y) \to D^+(X)$ such that

$$Hom_{D^+(Y)}(Rf_!\mathcal{F}, \mathcal{G}) \cong Hom_{D^+(X)}(\mathcal{F}, f^!\mathcal{G})$$

Well-generated triangulated categories

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- there is a cardinal α, s.t. the the objects of S are α-small, i.e.
- every map $s \to \coprod_{i \in I} x_i$ factors through some $\coprod_{j \in J} x_j$, s.t. $|J| < \alpha$

Brown representability

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- Let $\mathcal{T}_2$ be an arbitrary triangulated category
- A triangulated functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ admits a right adjoint $G : \mathcal{T}_2 \to \mathcal{T}_1$ if and only if it preserves coproducts

Brown and Verdier

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- Moreover $D(X)$ is well-generated since $Sh_k(X)$ is a Grothendieck abelian category
- As a consequence we obtain a right adjoint $f^! : D(Y) \to D(X)$

Verdier and bounded complexes

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- $\tau^{\leq i} f^! \mathcal{F} = 0 \Rightarrow f^! \mathcal{F} \in D^+(X)$

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- base change
- constructibility

Base Change 1

Let the following be a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Base Change 1

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Then we have an isomorphism of functors $g^* \circ Rf_! \cong Rf'_! \circ g'^*$

Base Change 2

Using Yoneda's lemma and the adjunction property of $Rf_!$ and $f^!$, we obtain

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an isomorphism of functors $f^! \circ Rg_* \cong Rg'_* \circ f'^!$

Local Verdier duality

- Given sheaves $\mathcal{F}$, $\mathcal{G}$ there is a natural morphism $f_* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(f_! \mathcal{F}, f_! \mathcal{G})$

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- $Rf_*R\mathcal{H}om(\mathcal{F}, f^!\mathcal{G}) \to R\mathcal{H}om(Rf_!\mathcal{F}, Rf_!f^!\mathcal{G})$
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- Using $Rf_!f^!\mathcal{G} \to \mathcal{G}$ we get
- $\phi : Rf_*R\mathcal{H}om(\mathcal{F}, f^!\mathcal{G}) \to R\mathcal{H}om(Rf_!\mathcal{F}, \mathcal{G})$

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Proof:

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- We have to check that for every open subset $U \subset Y$ the map $R\Gamma(U, \phi)$ is an iso:

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- $\mathcal{H}^i(R\Gamma(U, Rf_* R\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}))) \cong$ $Hom_{f^{-1}(U)}(\mathcal{F}|_{f^{-1}(U)}, f^!\mathcal{G}[i]|_{f^{-1}(U)}) \cong$

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- $Hom_U(R(f|_U)_! \mathcal{F}|_U, \mathcal{G}[i]|_U) \cong \mathcal{H}^i(R\Gamma(U, R\mathcal{H}om(Rf_! \mathcal{F}, \mathcal{G})))$

Local Verdier duality

Analogously we can show that

$$f^! R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong R\mathcal{H}om(f^*\mathcal{F}, f^!\mathcal{G})$$

Duality functor

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- The dualizing sheaf is the complex $\omega_X := a_X^! \mathbb{Q}$
- The Verdier dual is the functor $\mathcal{D}_X := R\mathcal{H}om(-, \omega_X)$
- e.g. for $X = \bullet$, $(\mathcal{D}_X \mathcal{F})^i = (\mathcal{F}^{-i})^*$

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$$f^! R\mathcal{H}om(-, a_Y^! \mathbb{Q}) \cong R\mathcal{H}om(f^* -, f^! a_Y^! \mathbb{Q}) \cong R\mathcal{H}om(f^* -, a_X^! \mathbb{Q})$$

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Proof:
$$Rf_* R\mathcal{H}om(-, a_X^! \mathbb{Q}) \cong Rf_* R\mathcal{H}om(-, f^! a_Y^! \mathbb{Q}) \cong R\mathcal{H}om(Rf_! -, a_Y^! \mathbb{Q})$$

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- the distinguished triangle

$$i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F} \xrightarrow{[1]} i_! i^! \mathcal{F}$$

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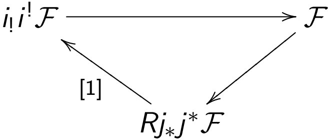

- where $j : U \to X$ is an open immersion and $i : Z \to X$ a closed immersion, and $U \cup Z = X$

Biduality

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As a consequence

- $f^! \cong \mathcal{D}_X f^* \mathcal{D}_y$
- $Rf_! \cong \mathcal{D}_Y Rf_* \mathcal{D}_X$
- in particular the functors $(Rf_*, f^*, Rf^!, f^!)$ descend to the bounded constructible categories D_c^b

The functor $f^!$

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Proposition

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For $U \subset X$ open such that $Z \subset U$ closed, set

$$\Gamma_Z(U, \mathcal{F}) := \ker(\mathcal{F}(U) \to \mathcal{F}(U \backslash Z))$$

For any $V \subset U$ open containing Z as a closed subset $\Gamma_Z(U, \mathcal{F}) \xrightarrow{\sim} \Gamma_Z(V, \mathcal{F})$ so define $\Gamma_Z(X, \mathcal{F})$ as $\Gamma_Z(U, \mathcal{F})$ for any such U.

$$\Gamma_Z(\mathcal{F}) : U \mapsto \Gamma_{Z \cap U}(U, \mathcal{F})$$

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Hence:

$$Hom_{D^+(X)}(Ri_!\mathcal{F}, \mathcal{G}) \cong Hom_{D^+(X)}(Ri_!\mathcal{F}, R\Gamma_Z(\mathcal{G}))$$
$$\cong Hom_{D^+(Z)}(\mathcal{F}, i^* \circ R\Gamma_Z(\mathcal{G}))$$

Hence $i^!(\cdot) \simeq i^* \circ R\Gamma_Z(\cdot)$ due to uniqueness of adjoints.

The functor $f^!$

Corollary

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Corollary

Let $j : U \hookrightarrow X$ be the inclusion of an open subset of X. Then $j^!(\cdot) = j^*(\cdot)$.

Deriving the equality $\Gamma_U = j_* \circ j^*$ gives $R\Gamma_U = Rj_* \circ j^*$.

Since $j^* \circ Rj_* = \mathbf{Id}$ the previous proposition gives $j^!(\cdot) = j^*(\cdot)$

Proposition

There is a natural morphism of functors from $D^+(Y) \times D^+(Y) \to D^+(X)$:

$$f^!(\cdot) \otimes^L_{\underline{k}_X} f^*(\cdot) \to f^!\left(\cdot \otimes^L_{\underline{k}_Y} \cdot\right)$$

Let $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2 \in D^+(Y)$.

$$Hom_{D^+(X)}(f^!\mathcal{F}_1 \otimes^L_{\underline{k}_X} f^*\mathcal{F}_2, f^!\mathcal{F}) \cong$$

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Let $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2 \in D^+(Y)$.

$$Hom_{D^+(X)}(f^!\mathcal{F}_1 \otimes^L_{\underline{k}_X} f^*\mathcal{F}_2, f^!\mathcal{F}) \cong$$
$$Hom_{D^+(Y)}(Rf_! \left(f^!\mathcal{F}_1 \otimes^L_{\underline{k}_X} f^*\mathcal{F}_2 \right), \mathcal{F}) \cong$$

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There is a natural morphism of functors from $D^+(Y) \times D^+(Y) \to D^+(X)$:

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Set $\mathcal{F} := \mathcal{F}_1 \otimes^L_{\underline{k}_Y} \mathcal{F}_2$ and take the image of

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The functor $f^!$

Here, we have used the *Projection formula*

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Definition

Call $f : X \to Y$ a *topological submersion* of *fiber dimension n* if, for each $x \in X$, there is an open neighbourhood U of x such that $V = f(U)$ is open in Y and the restriction $f|_U$ is topologically equivalent to the projection $V \times \mathbb{R}^n \to V$.

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Poincaré duality

Now assume $a_X : X \to \bullet$ is a topological submersion, that is, X is an n-dimensional topological manifold.

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