Constructible Sheaves and their derived category

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A first example

Let $\pi : E \to X$ be a vector bundle.

Sections

For every open $U \subset X$ consider

$$\Gamma_E(U) := \left\{ s \text{ in } \begin{array}{ccc} & & E \\ & \overset{s}{\nearrow} & \downarrow \pi \\ U & \hookrightarrow & X \end{array} \right\}$$

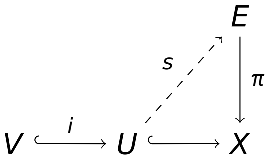
Restriction

Restriction

For $i : V \subset U$ we get a morphism

$$i^* := \mathrm{res}^V_U : \Gamma_E(V) \longrightarrow \Gamma_E(U)$$

This morphism comes from

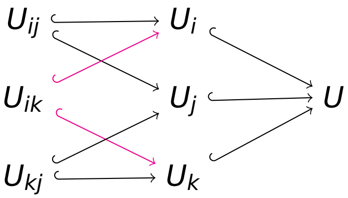


Glueing

Descent

Let $\mathcal{U}$ be a covering of an open $U \subset X$. Define

$$\mathrm{Desc}_{\mathcal{U}}(\Gamma_E) := \left\{ (x_i)_I \in \prod_{U_i \in \mathcal{U}} \Gamma_E(U_i) \;\middle|\; x_i|_{U_{ij}} = x_j|_{U_{ij}} \right\}$$



- We get an isomorphism of vector spaces

$$\mathrm{desc}_{\mathcal{U}} : \Gamma_E(U) \to \mathrm{Desc}_{\mathcal{U}}(\Gamma_E).$$

Presheaves

Presheaves

The category of **Presheaves on** X (with values in a category C) is the functor category

$$\mathrm{PSh}(X, C) := \mathrm{Cat}\left(\mathrm{Open}(X)^{\mathrm{op}}, C\right).$$

- The category of presheaves inherits many properties of C.

Presheaves

Limits

- If C admits limits or colimits, so does $\mathrm{PSh}(X, C)$.
- If C is **abelian**, so is $\mathrm{PSh}(X, C)$

Tensor Products

- Monoidal structure $\otimes : C \times C \to C$ induces $\otimes$ for $\mathrm{PSh}(X, C)$.
- Presheaves inherit an internal Hom:

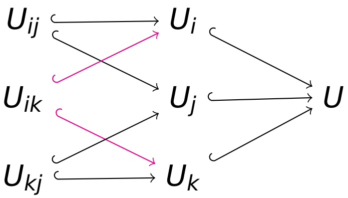
$$(- \otimes F) \dashv \underline{\mathrm{Hom}}(F, -).$$

Glueing

Descent

For any presheaf $\mathcal{F}$ define as above

$$\mathrm{desc}_{\mathcal{U}} : \mathcal{F}(U) \to \mathrm{Desc}_{\mathcal{U}}(\mathcal{F})$$



- For Set-valued presheaves:

$$\mathrm{Desc}_{\mathcal{U}}(\mathcal{F}) := \left\{ x \in \prod_{U_i \in \mathcal{U}} \Gamma_E(U_i) \;\middle|\; x_i|_{U_{ij}} = x_j|_{U_{ij}} \right\}$$

- In the general case:

$$\mathrm{Desc}_{\mathcal{U}}(\mathcal{F}) := \varprojlim_{\overline{\mathcal{U}}} \mathcal{F}$$

Sheaves

Definition

A presheaf is called a **sheaf** if $\text{desc}_{\mathcal{U}}$ is an isomorphism for every covering $\mathcal{U}$ of an open U. We denote the category by

$$\text{Sh(X,C)}.$$

In other words:

If we know $\mathcal{F}$ on a covering of an open subset $U \subset X$, we can reconstruct $\mathcal{F}(U)$.

Examples

Continuous functions

Let X be a topological space. We get a sheaf

$$\mathcal{C}_X : U \longmapsto \mathcal{C}(U, \mathbb{R}).$$

The structure sheaf

Let X be a variety. Then $\mathcal{O}_X$ is a sheaf for the Zariski topology.

Abelian sheaves

- Abelian presheaves form an abelian category.
- What about the category of abelian sheaves?

Too bad!

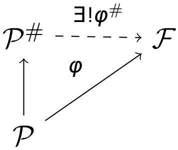
For a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ the cokernel in PSh(X) need not be a sheaf.

Still true

- Abelian sheaves form an abelian category.

Sheafification

We want to turn a presheaf $\mathcal{P}$ into a sheaf $\mathcal{P}^\#$ in a minimal way. We want



In other words

We want a left adjoint $(-)^\#$ to the forgetful functor

$$\mathrm{Sh}(X, C) \subset \mathrm{PSh}(X, C).$$

We will call it **sheafification**.

From spaces to sheaves

From spaces to sheaves

For $E \to X$ consider the sheaf of sections

$$\Gamma_E : U \mapsto \left\{ \begin{array}{ccc} & & E \\ & \overset{s}{\nearrow} & \downarrow \\ U & \hookrightarrow & X \end{array} \right\}.$$

This constitutes a functor $\mathrm{Top}/X \to \mathrm{Sh}(X, \mathrm{Set})$.

- We want to reverse this process, namely: **We want an adjoint!**

From presheaves to étalé spaces

Definition

Let $\mathcal{F} \in \mathrm{PSh}(X, \mathrm{Set})$ and $x \in X$. The **stalk** of $\mathcal{F}$ at x is

$$\mathcal{F}_x := \mathrm{colim}_{\mathcal{U}(x)}(\mathcal{F}).$$

From presheaves to spaces

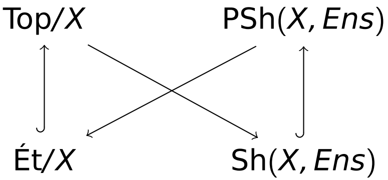
Let $\mathcal{F} \in \mathrm{PSh} X$. Consider

$$\mathrm{t}(\mathcal{F}) := \left(\coprod_{x \in X} \mathcal{F}_x \right) \to X$$

with an appropriate topology. This space is **étalé** over X

The situation

- Let's summarize the situation:



Theorem

This establishes an equivalance of categories

$$\text{Ét}/X \cong \text{Sh}(X, Ens).$$

An explicit formula for sheafification

Hands on!

Let $\mathcal{P}$ be a presheaf of sets. Then the following is a sheafification of $\mathcal{P}$:

$$\mathcal{P}^{\#}(U) := \left\{ x \in \prod_{u \in U} \mathcal{F}_u \;\middle|\; \begin{array}{c} \forall p \in U \;\; \exists U_p \in \mathcal{U}(p)_{\subset U} \\ \exists s_p \in \mathcal{F}(U_p) : \\ \forall q \in U_p : (s_p)_q = x_q \end{array} \right\}$$

Abelian Sheaves, pt. II

Remember

For a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ the cokernel in PSh(X) need not be a sheaf.

Solution

Replace the cokernel by its sheafification to see that $\mathrm{Sh}(X, \mathfrak{A})$ is abelian for abelian $\mathfrak{A}$.

Sheaves of modules

Sheaves of rings

A sheaf of rings $\mathcal{R}$ on X is the same as a ring-object in the category of sheaves:

$$\begin{array}{ccc} \mathcal{R} \otimes \mathcal{R} & \to & \mathcal{R} \\ \underline{\mathbb{Z}} & \to & \mathcal{R} \end{array}$$

Sheaves of modules

A sheaf of modules $\mathcal{M}$ on X for a sheaf of rings is given by

$$\mathcal{R} \otimes \mathcal{M} \to \mathcal{R}$$

.

Abelian Sheaves, pt. III

Same old

Abelian groups are nothing but $\mathbb{Z}$-modules.

Same old

Abelian sheaves on X are nothing but $\underline{\mathbb{Z}}_X$-modules.

Direct image of presheaves

- Let $f : X \to Y$ be continuous. We get

$$f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X).$$

This induces the precomposition functor:

$$f_* : \begin{cases} \mathrm{PSh}(X) & \to & \mathrm{PSh}(Y) \\ f_*\mathcal{F}(V) & = & \mathcal{F}(f^{-1}(V)) \end{cases}$$

We are lucky

Direct image of sheaves

$$\mathcal{F} \text{ sheaf } \Rightarrow f_*\mathcal{F} \text{ sheaf}$$

Inverse image - first try

- Is this process invertible?
- Let's try

$$f^*\mathcal{G}(U) := \mathcal{G}(f(U))$$

Too bad!

The set $f(U)$ need not be open; $\mathcal{G}(f(U))$ need not be defined.

Solution

Idea: Approximate $f(U)$ by open subsets:

$$[f(U)] := \left\{ V \subset Y \text{ open} \;\middle|\; f(U) \subset V \right\}$$

Inverse image

We define the inverse image of a presheaf $\mathcal{G} \in \mathrm{PSh}(Y)$ to be

$$f^{-1}\mathcal{G} := \left(U \mapsto \operatorname{colim}_{[f(U)]} \mathcal{G} \right).$$

Lemma

We have $f^{-1} \dashv f_*$.

Too bad!

$$\mathcal{G} \text{ sheaf } \not\Rightarrow f^{-1}\mathcal{G} \text{ sheaf}$$

Definition

The inverse image of sheaves is

$$f^* := (-)^\# \circ f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X).$$

- As composition preserves adjoints we get

$$f^* \dashv f_*.$$

The situation so far

$$\mathrm{Sh}(X, C) \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Sh}(Y, C)$$

Functoriality

The assignment $X \mapsto \mathrm{PSh}X$ is functorial (up to isomorphism).

- $\mathrm{id}_* \cong \mathrm{id}$
- $f_* g_* \cong fg_*$
- The same holds for f^*.

Why Sheaf Cohomology?

We aim to understand the structure of spaces and extra structure on them.

Motivation

- 0-th cohomology counts number of connected components
  Compare to singular cohomology, which counts number of path components.
- will help to understand the first fundamental group $\pi_1(X)$
- more flexible than singular cohomology
- measure exactness of functors, for example Γ
- ...

Global Sections

Definition

For a sheaf $\mathcal{F}$ on X consider the **global sections**

$$\Gamma(X, F) := F(X).$$

This consitutes a functor $\Gamma(X, -) : \mathrm{Sh}(X, C) \to C$

Remark

This is the same functor as pt_*.

- Here $\mathrm{pt} : X \to \mathrm{pt}$
- Remember

$$\begin{array}{rcl} \mathrm{PSh}(X) & \to & \mathrm{PSh}(Y) \\ f_*\mathcal{F}(V) & = & \mathcal{F}(f^{-1}V) \end{array}$$

Exactness

Fun ahead!

The functor $\Gamma(X, -)$ is only left exact.

Remark

To deal with this, we have to consider the right derived functor. We have to ascend to $D(\mathcal{S}\mathrm{h}(X))$ the

derived category of sheaves.

More generally, we will study $D(\mathcal{R}-\mathcal{M}\mathrm{od})$.

Exactness and stalks, pt. I

Lemma

A sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact **iff** it is exact on every stalk:

$$0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0.$$

Remark

A point $x \in X$ is nothing but $x : \mathrm{pt} \to X$ and we have

$$x^* \mathcal{F} = \mathcal{F}_x$$

Exactness and stalks, pt. II

Lemma

The functor f^* is exact.

Proof

- By functoriality:

$$(f^*\mathcal{F})_x \xrightarrow{\sim} \mathcal{F}_{f(x)}.$$

- So for an short exact sequence we get:

$$\begin{array}{ccccc} \mathcal{F}'_{f(x)} & \hookrightarrow & \mathcal{F}_{f(x)} & \twoheadrightarrow & \mathcal{F}''_{f(x)} \\ \uparrow & & \uparrow & & \uparrow \end{array}$$

Derived functors

Tool

$\mathcal{S}\mathrm{h}(X)$ has enough injective objects.
Hence, there is a way to calculate right derived functors.

Example: The Constant sheaf

- We are especially interrested in a certain class of sheaves.
- Again, consider $\mathrm{pt} : X \to \mathrm{pt}$. This gives

$$\mathrm{pt}^* : \mathbb{Z}\text{-mod} \to \mathrm{Sh}(X, \mathbb{Z}).$$

Definition

A sheaf of the form $\underline{A}_X := \mathrm{pt}^* A$ is called **constant sheaf**.

- For locally connected spaces, we have $(\underline{A}_X)_x = A$.
- The sheaf $\underline{A}_X$ is the sheafification of the constant presheaf $U \mapsto A$.

Sheaf cohomology

Defininition

The i-th cohomology of X with values in a sheaf $\mathcal{F}$ is defined to be

$$H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F}) = H^i(Rpt_*(\mathcal{F}))$$

Remark

This is often applied to

$$\mathcal{F} = \underline{A}_X$$

where $A = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \ldots$.

Sheaf Cohomology behaves "good"

Theorem

For a nice space X many cohomology theories agree:

$$\check{H}^\bullet(X, \underline{\mathbb{Z}}_X) = H^\bullet_{\text{sing}}(X, \underline{\mathbb{Z}}_X) = R^\bullet pt_* \underline{\mathbb{Z}}_X$$

For example, the first equation holds, if X is homotopy equivalent to a CW-complex,
whereas the second, if X is paracompact and locally contractible.

The Long Exact Sequence

Given an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of sheaves on X, we obtain a long exact sequence by the formalism of derived categories.

$$\dots \to H^n(X, \mathcal{F}') \to H^n(X, \mathcal{F}) \to H^n(X, \mathcal{F}'') \to H^{n+1}(X, \mathcal{F}') \to \dots$$

Base change

Consider a commutative square of spaces

$$\begin{array}{ccc} W & \xrightarrow{h'} & Y \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{h} & Z \end{array}$$

- Contemplating for a while we find a transformation

$$h^* \ast \circ v_\ast \to v'_\ast \circ h'^*$$

Base change

In some lucky cases, this is an isomorphism.

Functoriality

Let $f : X \to Y$. Consider the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathrm{pt} \end{array}$$

- Base change induces

$$\Gamma(Y, -) = \Gamma(X, f^*(-)).$$

- We specialize to

$$H^\bullet(Y, \underline{\mathbb{Z}}_Y) \to H^\bullet(X, \underline{\mathbb{Z}}_X).$$

Flabby sheaves

Definition

$\mathcal{F} \in \mathcal{S}\mathrm{h}(X)$ is flabby or flasque, if it $\mathcal{F}(i)$ is an epimorphism for all $i \in \mathrm{Open}(U, V)$,
iff restriction from X induces epimorphisms.

Example

Any constant sheaf on an irreducible variety.
Remember that $\underline{A}_X(U) = \bigoplus A = A$,
where the sum runs over the number of connection components of U, which is one in this case.

Vanishing of Cohohmology

Problem/Proposition

Let $\mathcal{F} \in \mathcal{S}\text{h}(X)$ be flabby, then $H^i(X, \mathcal{F}) = 0 \ \ \forall i \geq 1$.

Idea of proof

- Take a short exact sequence $\mathcal{F} \to \mathcal{I} \to G$, with $\mathcal{I}$ injective.
- Hence $\mathcal{G}$ is flabby.
- Now proceed by induction using the long exact sequence.

Conclusion

The Zariski topology is too coarse to obtain useful results with sheaf cohomology (with constant coefficients).
Open problem: Construct cohomology theory using the Zariski topology.
Solution: Change topology!
Fact: There is good evidence that the following construction, yields the ”correct” cohomology theory.

Analytic topology of a variety

Definition

Let X be a variety. The associated analytic space X^{an} is obtained as follows:
Cover X by open affine subsets and take the topology induced from the standard topology of $\mathbb{C}^n$.

The Derived Category of Interest

For an algebraic variety X we denote
$D(X) := D(\underline{\mathbb{C}}_{X^{an}} - \mathcal{M}\mathrm{od})$.

$\pi_1(X)$ and local systems

Definition

A local system on a variety, is a locally free $\underline{\mathbb{C}}_{X^{an}} - \mathcal{M}od$ of finite rank.

Why care: For X paracompact, Hausdorff, path-connected and locally 1-connected we have:

$$\mathcal{L}oc(X) \xrightarrow{\sim} \mathcal{R}ep - \pi_1(X, x_0)$$

Particularly, $\mathcal{L}oc(X) \subset \underline{\mathbb{C}}_X - \mathcal{M}od$ is an abelian subcategory.
The assumptions hold for X^{an}, but not for X.
This gives another reason for the analytification, as there are no non constant locally constant sheaves on irreducible varieties.

Why constructible sheaves?

As seen in the motivation, (locally) constant sheaves contain a lot of information about varieties.
We obtain them by derived functors, hence we

Want

A nice subcategory of $\mathrm{D}(X)$ that contains:

- $\mathcal{L}\mathrm{oc}(X)$
- Extensions by zero (see next talk) of $\mathcal{L}\mathrm{oc}$
- Extensions of the above in $\underline{\mathbb{C}}_X - \mathcal{M}\mathrm{od}$
- Rpt_* computes cohomology, therefore include image of Rf_* for morphisms $f : X \to Y$

The Leray-Serre Spectral Sequence

For $f : X \to Y$ a continuous map, we have $Rpt_* = Rpt_* \circ Rf_*$. This gives rise to the spectral sequence $H^p(Y, R^q f-) \Rightarrow H^{p+q}(X, -)$

Example

Let $\pi : E \to B$ be a locally trivial topological fibration with fibre F and $\mathcal{L} \in \mathcal{L}oc(E)$, then:

$$\mathcal{H}^i(R\pi_* \mathcal{L})|_U \cong \underline{H^i(F, \mathcal{L}|_F)}_U$$

For $U \subset B$ open, simply connected and trivializing. Hence, the cohomology sheaves of $R\pi_* \mathcal{L}$ are local systems.

Stratification and constructible sheaves, pt.I

Definition

A locally finite paritition $\mathcal{P} = (X_i)_{i \in I}$ of a variety X into connected and locally closed, smooth subvarieties is a stratification (the sets X_i are called the strata), if $\bar{X}_i$ is a union of strata.

Stratification and constructible sheaves, pt.II

Definition

- A sheaf $\mathcal{F} \in \underline{\mathbb{C}}_X - \mathcal{M}od$ with finite dimensional stalks is constructible, if there exists a stratification $\mathcal{P}$, such that $\mathcal{F}|_{X_i} \in \mathcal{L}oc(X_i)$ for all $i \in I$
- A complex $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$ is constructible, if all its cohomology sheaves are constructible.
  We denote $D^b_c(X)$ the full triangulated subcategory of constructible complexes.

Remarks on the Construction

- Why not first constructible sheaves and derive then?
- **Fact:** Constructible sheaves are abelian, but do not have enough injectives.

Theorem

There is a natural functor $D^b(\underline{\mathbb{C}}_{X^{an}} - \mathcal{M}\mathrm{od}_{constr}) \to D^b_c(X)$. If X is a complex algebraic variety, then it is an equivalence, but in general not!

Derived Tensor Produkt and internal Hom

Want: derived $\otimes$ and $\mathcal{H}$om, as well.

- $R\,\mathcal{H}\mathrm{om}(-,-)$ is constructed analogously to $R\,\mathrm{Hom}(-,-)$.
- $\otimes$ is right exact. So we need projective resolutions to derive it.
  But they do not exist, therefore take flat resolutions. We obtain the left derived bifunctor $\otimes^L$.
- Both are only defined on subcategories, but certainly as follows:

$$D^b(X)^{(op)} \times D^b(X) \to D^b(X)$$

Theorem

The category $D^b_c(-)$ is preserved by the following functors:

- Let $f : X \to Y$ be a morphism of algebraic varieties. Then $Rf^{-1} : D^b_c(Y) \to D^b_c(X)$ and $Rf_* : D^b_c(X) \to D^b_c(Y)$
- $R\mathcal{H}om$ and $\otimes^L$ restrict to functors $D^b_c(X)^{(op)} \times D^b_c(X) \to D^b_c(X)$