Combinatorics of Weyl Groups and Kazhdan Lusztig Polynomials

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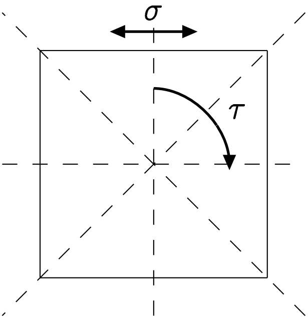
We consider the dihedral group of order 8. This group can be thought of as the symmetry group of the square



Three rotational symmetries: $\tau,\ \tau^2,\ \tau^3$

Four reflection symmetries: $\sigma,\ \tau\sigma,\ \tau^2\sigma,\ \tau^3\sigma$

We can express this group using a set of generators and relations

$$\langle \sigma, \tau \mid \tau^4 = \sigma^2 = 1, \sigma\tau\sigma = \tau^{-1} \rangle.$$

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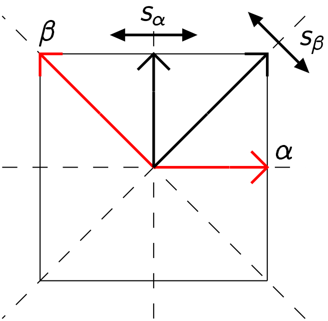


Three rotational symmetries:

$s_\beta s_\alpha$, $s_\alpha s_\beta s_\alpha s_\beta$, $s_\alpha s_\beta$

Four reflection symmetries:

s_α, s_β, $s_\beta s_\alpha s_\beta$, $s_\alpha s_\beta s_\alpha$

We can express this group using a set of generators and relations

$$\langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = 1, (s_\alpha s_\beta)^4 = 1 \rangle.$$

Definition

A group G is called a Coxeter group if it is isomorphic to a group given by a presentation of the form

$$\langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ii} = 1$ and $m_{ij} \geqslant 2$ for all $i \neq j$. If no relation occus for a pair s_i, s_j we let $m_{ij} = \infty$. We usually denote a set of generators by S.

The general study of Coxeter groups can be split in to the study of finite and infinite Coxeter groups. All finite Coxeter groups are finite real reflection groups. We will be interested in a large subclass of these groups known as Weyl groups.

Example

Any dihedral group D_{2m} is a finite Coxeter group. We know D_{2m} is the symmetry group of a regular m-gon centred at the origin in $\mathbb{R}^2$. Let s_1, s_2 be two reflection symmetries of the m-gon adjacent by an angle of π/m then we have

$$D_{2m} \cong \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1 s_2)^m = 1 \rangle.$$

How do we generalise reflections to higher dimensional vector spaces?

Definition (or Recollection)

If V is a real Euclidean vector space then a reflection in V is a map $s_\alpha : V \to V$ which sends $0 \neq \alpha \in V$ to $-\alpha$ and fixes pointwise the hyperplane $H_\alpha = \{\lambda \in V \mid \lambda \perp \alpha\}$ orthogonal to α.

Notation

From now on V will denote a finite dimensional real Euclidean vector space with Euclidean inner product $(\cdot,\cdot) : V \times V \to \mathbb{R}$. If $\alpha, \beta \in V$ then we write $\langle \beta, \alpha \rangle$ for the value $2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$.

Definition

A set of non-zero vectors $\Phi \subset V$ is called a crystallographic root system for V if:

- Φ spans V,
- if $\alpha \in \Phi$ then $r\alpha \in \Phi \Rightarrow r = \pm 1$,
- if $\alpha \in \Phi$ then $s_\alpha \Phi = \Phi$.
- if $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Definition

Given a root system $\Phi \subset V$ we define $W := \langle s_\alpha \mid \alpha \in \Phi \rangle \leqslant O(V)$ to be the associated reflection group of Φ. We can easily describe the action of a reflection s_α on V by the formula

$$s_\alpha \lambda = \lambda - \langle \lambda, \alpha \rangle \alpha$$

for all $\alpha \in \Phi$ and $\lambda \in V$. If Φ is crystallographic then we call W a Weyl group.

Why Crystallographic?

Let Φ be crystallographic and consider $\mathbb{Z}\Phi \subset V$. If $\lambda \in \mathbb{Z}\Phi$ then $s_\alpha \lambda \in \mathbb{Z}\Phi$ for all $\alpha \in \Phi$. Hence $s_\alpha(\mathbb{Z}\Phi) \subseteq \mathbb{Z}\Phi$ and so W preserves the lattice $\mathbb{Z}\Phi$ of V.

Some quick observations and basic concepts

- Every root system has an expression as a disjoint union $\Phi = \Phi^+ \sqcup \Phi^-$ such that $\Phi^- = -(\Phi^+)$, $|\Phi^-| = |\Phi^+|$ and if α, $\beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$. Call Φ^+, (resp. Φ^-), a positive system, (resp. negative system).
- A subset $\Delta \subset \Phi$ is a simple system if it is a vector space basis of V and every root in Φ can be expressed as a linear combination of elements of Δ with either all positive or all negative coefficients. (Note: fixing a positive system uniquely determines a simple system and vice versa.)
- We call $|\Delta|$ the rank of Φ.
- If $\alpha = \sum_{\gamma \in \Delta} c_\gamma \gamma \in \Phi$ then we call $\sum_{\gamma \in \Delta} c_\gamma$ the height of α.

Some special facts about root systems:

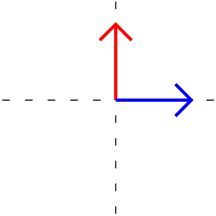
- W is generated by the reflections s_i associated to the simple roots $\alpha_i \in \Delta$ and has a Coxeter presentation.
- We define a length function $\ell : W \to \mathbb{N}$ such that $\ell(1) = 0$ and $\ell(w)$, for all $w \neq 1$, is the minimal number m such that w can be expressed as a product of m simple reflections.
- For a fixed simple system Δ there is a unique element $w_0 \in W$, known as the longest word, such that $w_0^2 = 1$ and $w_0(\Phi^+) = \Phi^-$.

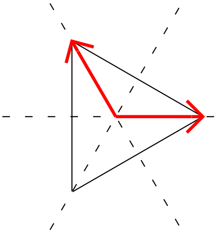
Some special facts about crystallographic root systems:

- We have $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3, 4\}$. Consequently the angle between two distinct roots must be $\pi/2$, $\pi/3$, $\pi/4$, $\pi/6$, $2\pi/3$, $3\pi/4$ or $5\pi/6$ and also $m_{ij} \in \{2, 3, 4, 6\}$ where $(s_i s_j)^{m_{ij}} = 1$.
- There are at most two distinct root lengths in Φ with the ratio $|\alpha|^2 : |\beta|^2$ being $1 : 1$, $2 : 1$ or $3 : 1$.

When Φ is of rank 2 we know that a simple system forms a basis for $\mathbb{R}^2$. Hence it is easy to draw pictures of the rank 2 crystallographic root systems.

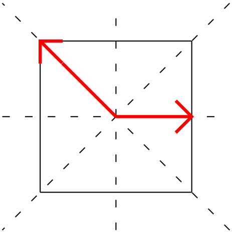
$A_1 \times A_1$

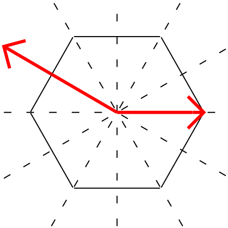




A_2

B_2



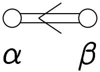


G_2

We want a nice way to determine all crystallographic root systems. To do this we encode the information of Φ in a graph called the Dynkin diagram of Φ. This is a graph whose nodes are labelled by simple roots. We then connect the nodes by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges for all simple roots α_i, α_j.

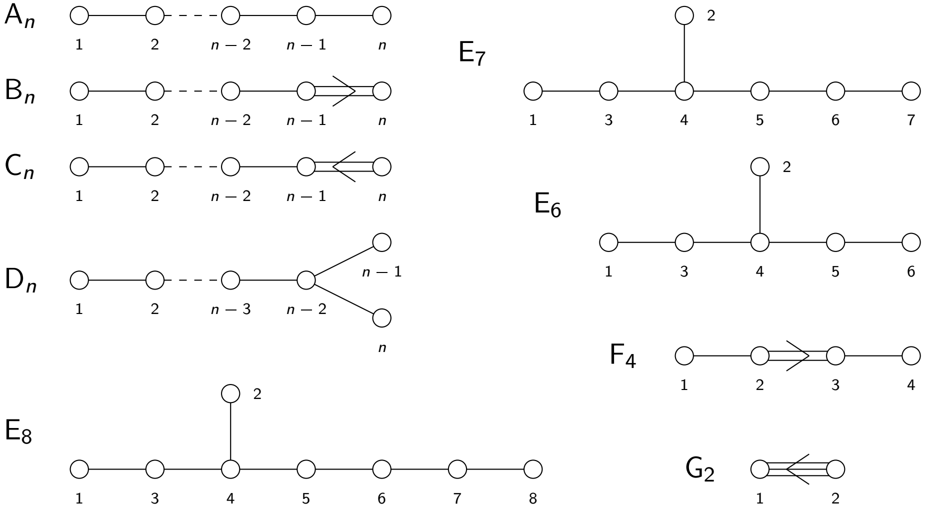
Example

Consider the dihedral group D_8. The root system Φ affording D_8 as a reflection group has two simple roots $\{\alpha, \beta\}$ such that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 2$. Choosing α to be the short root we have the Dynkin diagram of Φ is given by



We call Φ irreducible if its Dynkin diagram is connected.

The Dynkin Diagrams of the irreducible crystallographic root systems.



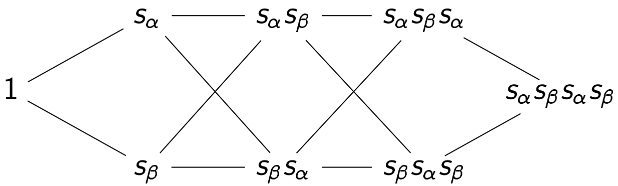
We would like to consider a way to partially order the elements of W using the length function. There are in fact many ways to put a partial order on W. However the most useful for our purposes is known as the Bruhat order.

Definition

Let Φ be a root system with set of reflections $T = \{s_\alpha \mid \alpha \in \Phi\}$, (not just simple reflections), and associated reflection group W. For all $w, w' \in W$ we write $w' \to w$ if there exists $t \in T$ such that $w = w't$ with $\ell(w) > \ell(w')$. Then define $w < w'$ if there is a sequence $w' = w_0 \to w_1 \to \cdots \to w_m = w$. We call the relation $<$ the Bruhat order.

Let $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta)\}$ be the root system of type B_2, such that α is the short root. We have already seen that the associated Weyl group generated by the reflections s_α, s_β is the dihedral group D_8. There are four reflections associated to these roots, namely $T = \{s_\alpha, s_\beta, s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta\}$.

We now consider the Hasse diagram of the Bruhat ordering on D_8.



Definition

Let $A := \mathbb{Z}[q, q^{-1}]$, (Laurant polynomial ring with indeterminate q), and let $W = W(S)$ be a Coxeter group. The (generic) Hecke algebra associated with W is the free A-module H with basis T_w, $(w \in W)$, satisfying the relations

$$T_s T_w = T_{sw} \qquad \text{if } \ell(sw) > \ell(w),$$
$$T_s^2 = (q-1)T_s + qT_1.$$

This gives rise to a unique associative algebra structure.
Sometimes we replace A by $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ to help with computations.

Specialisation

Let $\lambda \in \mathbb{C}$. Consider the tensor product

$$\mathbb{C} \otimes_A H$$

where $\mathbb{C}$ is an A-module via $f \cdot \alpha = f(\lambda)\alpha$ for $f \in A, \alpha \in \mathbb{C}$.
This is a $|W|$-dimensional $\mathbb{C}$-algebra.
E.g. when $\lambda = 1$, we get $\mathbb{C} \otimes_A H \cong \mathbb{C}[W]$. In fact the same is true when λ has infinite order in $\mathbb{C}$.
When λ is a root of unity, it may be that $\mathbb{C} \otimes_A H$ is non-semisimple.
These specialied algebras occur naturally in the representation theory of finite groups of Lie type (Iwahori's theorem).

From now on we only treat the generic case. Recall the relations:

$$T_s T_w = T_{sw} \qquad \text{if } \ell(sw) > \ell(w),$$
$$T_s^2 = (q-1)T_s + qT_1.$$

Theorem

For all $w \in W$, T_w is invertible and its inverse is given by

$$(T_{w^{-1}})^{-1} = (-1)^{\ell(w)} q^{-\ell(w)} \sum_{x \leq w} (-1)^{\ell(x)} R_{x,w}(q) T_x,$$

where $R_{x,w}(q) \in \mathbb{Z}[q]$ is a polynomial of degree $\ell(w) - \ell(x)$ in q (an R-polynomial), and $R_{w,w}(q) = 1$.

Algorithm for Computing $R_{x,w}$

- Let $x, w \in W$. $R_{x,w} = 0$ unless $x \leq w$, while $R_{w,w} = 1$ for all $w \in W$, so assume $x < w$.
- Proceed by induction: assume $R_{y,z}$ known for $\ell(z) \leq \ell(w)$, then fix $s \in S$ such that $sw < w$.

1. If $sx < x$ then $R_{x,w} = R_{sx,sw}$.
2. If $sx > x$ then $R_{x,w} = (q-1)R_{x,sw} + qR_{sx,sw}$.

We define an involution $\iota : H \to H$ by extending additively

$$\iota : f(q) \mapsto f(q^{-1}) \qquad \text{for } f(q) \in A = \mathbb{Z}[q, q^{-1}],$$
$$\iota : T_w \mapsto (T_{w^{-1}})^{-1} \qquad \text{for } w \in W.$$

Theorem (Kazhdan-Lusztig (1979))

For each $w \in W$, there exists a unique element $C_w \in H$ with the following properties:

1. $\iota(C_w) = C_w$
2. $C_w = (-1)^{\ell(w)} q^{\frac{\ell(w)}{2}} \sum (-1)^{\ell(x)} q^{-\ell(x)} P_{x,w}(q^{-1}) T_x$ *(sum over $x \leq w$), where $P_{w,w}(q) = 1$ and $P_{x,w}(q) \in \mathbb{Z}[q]$ has degree $\leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$ if $x < w$.*

These C_w form a basis of H known as the Kazhdan-Lusztig basis.

Algorithm for computing $P_{x,w}$

Let $x, w \in W$.

- $P_{x,w} = 0$ unless $x \leq w$, while $P_{w,w} = 1$ for all $w \in W$, so assume $x < w$.
- Proceed by induction: assume $P_{y,w}$ known for $x < y \leq w$.
- Then the formula

$$q^{\ell(w)-\ell(x)} P_{x,w}(q^{-1}) - P_{x,w}(q) = \sum_{x<y\leq w} R_{x,y} P_{y,w}$$

yields $P_{x,w}$ uniquely.

Solving only basic linear relations on coeffiecients required for the latter.

Computational shortcuts for R-polynomials

- $R_{x,w}(1) = 0$ unless $x = w$.
- W finite $\Rightarrow R_{x,w} = R_{w_0 w, w_0 x}$ for all $x \leq w$.
- If $\theta : W \to W$ is an isomorphism, such that $\theta(S) = S$, then $R_{x,w} = R_{\theta(x),\theta(w)}$.

Computational shortcuts for K-L polynomials

- $P_{x,w}(0) = 1$ for all $x \leq w$.
- $0 \leq \ell(w) - \ell(x) \leq 2 \Rightarrow P_{x,w} = 1$.

Example (D_8)

- Let's compute R_{1,s_α}. We pick $s = s_\alpha$ so that we are in case 2. Then $R_{1,s_\alpha} = (q-1)R_{1,1} + qR_{s_\alpha,1} = q-1$.
- Now for $R_{1,s_\alpha s_\beta}$. We pick $s = s_\alpha$ so that we are in case 2 again. Then $R_{1,s_\alpha s_\beta} = (q-1)R_{1,s_\beta} + qR_{s_\alpha,s_\beta} = (q-1)^2$.
- Now for $R_{s_\alpha,s_\alpha s_\beta}$. We pick $s = s_\alpha$ but now we are in case 1. Then $R_{s_\alpha,s_\alpha s_\beta} = R_{1,s_\beta} = q-1$.

In fact it turns out that $R_{x,w}$ depends only on $\ell(w) - \ell(x)$:

$\ell(w) - \ell(x)$	$R_{x,w}$
0	1
1	$q-1$
2	q^2-2q+1
3	q^3-2q^2+2q-1
4	$q^4-2q^3+2q^2-2q+1$

Example (D_8 continued)

- Now for the K-L polynomials $P_{x,w}$. Recall that we need only consider $\ell(w) - \ell(x) = 3$ or 4.
- For $P_{1,s_\alpha s_\beta s_\alpha}$, the formula becomes

$$q^3 P_{x,w}(q^{-1}) - P_{x,w}(q) = \sum_{x<y\leq w} R_{x,y},$$

which yields that $P_{1,s_\alpha s_\beta s_\alpha} = 1$.
- Similarly we find that all K-L polynomials are 1 for this group.