Critical level representations of affine Kac–Moody algebras

Peter Fiebig

Emmy–Noether–Zentrum
Universität Erlangen–Nürnberg

Isle of Skye
May 2010

Affine Kac–Moody algebras

Let $\mathfrak{g}$ be a complex simple Lie algebra of finite dimension. We define

$$\begin{aligned} \widetilde{\mathfrak{g}} &= \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}], \text{ the } \textit{loop algebra}, \\ \widehat{\mathfrak{g}} &= \widetilde{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}D \text{ the affine Kac–Moody algebra.} \end{aligned}$$

The bracket is given by

$$\begin{aligned} [K, \cdot] &= 0 \\ [D, x \otimes t^n] &= nx \otimes t^n \\ [x \otimes t^n, y \otimes t^m] &= [x, y] \otimes t^{m+n} + n\delta_{m,-n}\kappa(x, y)K, \end{aligned}$$

where $\kappa \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ denotes the Killing-form.

Simple highest weight modules

We fix $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$, a Borel and a Cartan subalgebra. The corresponding affine Borel and Cartan subalgebras are

$$\widehat{\mathfrak{b}} := \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \oplus \mathbb{C}K \oplus \mathbb{C}D,$$
$$\widehat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

To any $\lambda \in \widehat{\mathfrak{h}}^\star = \mathrm{Hom}_{\mathbb{C}}(\widehat{\mathfrak{h}}, \mathbb{C})$ we associate the simple module $L(\lambda)$ with highest weight λ. We want to calculate its *character*

$$\mathrm{char}\ L(\lambda) = \sum_{\mu \in \mathfrak{h}^\star} \dim_{\mathbb{C}} L(\lambda)_\mu e^\mu.$$

Here, $L(\lambda)_\mu$ denotes the μ-eigenspace of the $\widehat{\mathfrak{h}}$-action.

Simple highest weight modules

The affine Weyl group $\widehat{\mathcal{W}}$ acts on $\widehat{\mathfrak{h}}^\star$. Let $\rho \in \widehat{\mathfrak{h}}^\star$ be an element with

$$\langle \rho, \alpha^\vee \rangle = 1$$

for each simple coroot $\alpha^\vee$. The *dot-action* of $\widehat{\mathcal{W}}$ on $\widehat{\mathfrak{h}}^\star$ is given by

$$w.\lambda = w(\lambda + \rho) - \rho.$$

It does not depend on the choice of ρ.

Definition

The *critical hyperplane* is

$$\{\lambda \in \widehat{\mathfrak{h}}^\star \mid \langle \lambda, K \rangle = \langle -\rho, K \rangle\}.$$

Known instances

The character of $L(\lambda)$ is known if

- λ is non-critical (Kashiwara & Tanisaki),
- if $\lambda \in \widehat{\mathcal{W}}.(-\rho + \overline{\rho})$ (Frenkel & Gaitsgory),
- if λ is critical and generic (Feigin & Frenkel),
- if λ is subgeneric (Arakawa & –).

Category $\mathcal{O}$

The category $\mathcal{O} \subset \widehat{\mathfrak{g}}$-mod contains all modules M with

- a semisimple $\widehat{\mathfrak{h}}$-action,
- a locally finite $\widehat{\mathfrak{b}}$-action.

We have an abstract block decomposition

$$\mathcal{O} = \prod_{\Lambda} \mathcal{O}_{\Lambda}.$$

We identify each index Λ with the set $\{\lambda \in \widehat{\mathfrak{h}}^{\star} \mid L(\lambda) \in \mathcal{O}_{\Lambda}\}$.

Blocks

We call $\mathcal{O}_\Lambda$ *critical* if Λ contains a critical weight. This is the case if and only if each weight in Λ is critical.
Let Λ be a block index and set

$$\widehat{\mathcal{W}}_\Lambda = \left\{ s_\alpha \;\middle|\; \begin{array}{c} \alpha \text{ is an affine real root with} \\ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for some/all} \\ \lambda \in \Lambda \end{array} \right\}$$

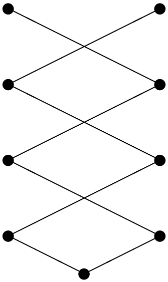
Theorem (Deodhar & Gabber & Kac, Kac & Kazhdan)

- *If Λ is non-critical, then $\Lambda = \widehat{\mathcal{W}}_\Lambda.\lambda$.*
- *If Λ is critical, then $\Lambda = \widehat{\mathcal{W}}_\Lambda.\lambda + \mathbb{Z}\delta$.*

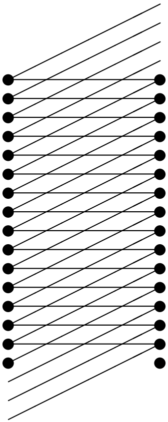
Here, $\delta \in \widehat{\mathfrak{h}}^\star$ is the smallest positive imaginary root. It is given by $\delta(D) = 1$, $\delta(\mathfrak{h} \oplus \mathbb{C}K) = 0$.

Block indices

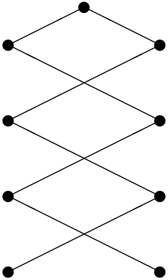
Negative level



Critical level



Positive level



Problems (?)

Let Λ be critical.

- Λ is not an orbit.
- There is no highest or lowest weight in Λ.
- The partial order on each orbit in Λ is not compatible with the Bruhat order.

We are going to address these problems.

The affine Weyl group and the set of alcoves

Let $\mathcal{A}$ be the set of alcoves and $A_e \in \mathcal{A}$ the base alcove. We identify

$$\begin{aligned} \widehat{\mathcal{W}} &\xrightarrow{\sim} \mathcal{A} \\ w &\mapsto w(A_e). \end{aligned}$$

- $\widehat{\mathcal{W}}$ carries the Bruhat order $\leq$.
- $\mathcal{A}$ carries the *generic Bruhat order* $\preceq$.
- The above map is not order preserving.

We now think of $\widehat{\mathcal{W}}$ and $\mathcal{A}$ as partially ordered sets being acted upon by simple reflections from the right. The above map intertwines these actions.

The affine Hecke algebra and its periodic module

We denote by $\widehat{\mathcal{H}} = \bigoplus_{w \in \widehat{\mathcal{W}}} \mathbb{Z}[v, v^{-1}] T_w$ the affine Hecke algebra. For any simple reflection we set $\underline{H}_s := vT_s + vT_e$. It satisfies

$$T_w \cdot \underline{H}_s = \begin{cases} v^{-1} T_{ws} + v^{-1} T_w, & \text{if } ws \leq w, \\ vT_{ws} + vT_w, & \text{if } w \leq ws. \end{cases}$$

We denote by $\widehat{\mathbf{M}} = \bigoplus_{A \in \mathcal{A}} \mathbb{Z}[v, v^{-1}] A$ the *periodic* $\widehat{\mathcal{H}}$-module. It satisfies

$$A_w \cdot \underline{H}_s = \begin{cases} A_{ws} + v^{-1} A_w, & \text{if } A_{ws} \succeq A_w, \\ A_{ws} + vA_w, & \text{if } A_w \succeq A_{ws}. \end{cases}$$

The Hecke algebra vs. its periodic module

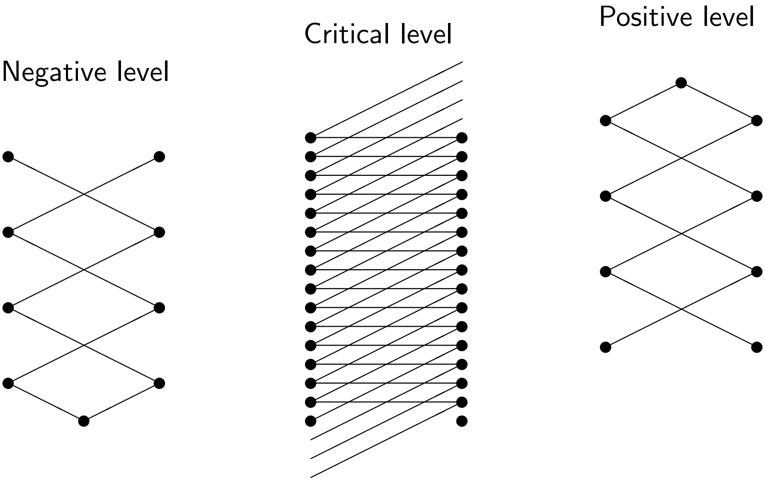
The Hecke algebra governs

- Iwahori-constructible simple perverse sheaves with $\mathbb{C}$-coefficients on the affine flag variety,
- Non-critical level representations of $\widehat{\mathfrak{g}}$.

The periodic module governs

- restricted representations of $\mathfrak{g}_k$, char $k \gg 0$ (conjecturally, char $k > h$),
- representations of the small quantum group at an l-th root of unity for $l > h$,
- critical level representations (conjecturally).

Block indices



Critical blocks

The simple highest weight module $L(\delta)$ is one-dimensional and invertible, i.e.

$$L(\delta) \otimes L(-\delta) = \mathbb{C}_{triv}.$$

We define the *shift functor*

$$\begin{aligned} T\colon \mathcal{O} &\to \mathcal{O} \\ M &\mapsto M \otimes L(\delta). \end{aligned}$$

It is an autoequivalence with inverse $T^{-1} = \cdot \otimes L(-\delta)$.

Observation

A block $\mathcal{O}_\Lambda$ is critical if and only if it is preserved by T, i.e. if and only if $T(\mathcal{O}_\Lambda) \subset \mathcal{O}_\Lambda$.

The graded center

Now we fix a critical block $\mathcal{O}_\Lambda$. For $n \in \mathbb{Z}$ we denote by $\mathrm{Mor}(\mathrm{id}, T^n)$ the set of natural transformations between the functors $\mathrm{id}, T^n \colon \mathcal{O}_\Lambda \to \mathcal{O}_\Lambda$. We define

$$\mathcal{A}_n := \left\{ z \in \mathrm{Mor}(\mathrm{id}, T^n) \,\middle|\, \begin{array}{c} \text{for all } M \in \mathcal{O}_\Lambda \text{ we have} \\ T^l(z^M) = z^{T^l M} \colon T^l M \to T^{n+l} M \end{array} \right\}$$

Then $\mathcal{A} := \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$ carries a canonical $\mathbb{C}$-algebra structure. It is commutative and associative and *HUGE*.

Restricted critical representations

Definition

We say that $M \in \mathcal{O}_\Lambda$ is restricted if $z^M = 0$ for all $z \in \mathcal{A}_n$, $n \neq 0$. We denote by $\overline{\mathcal{O}}_\Lambda \subset \mathcal{O}_\Lambda$ the subcategory of restricted modules.

Each $M \in \mathcal{O}_\Lambda$ has

- a largest restricted quotient $M \to \overline{M}$,
- a largest restricted submodule $\underline{M} \to M$.

Let $\Delta(\lambda)$ and $\nabla(\lambda)$ be the Verma and the dual Verma module associated to $\lambda \in \Lambda$. The *restricted* Verma and the *restricted* dual Verma module are

$$\overline{\Delta}(\lambda) \text{ and } \underline{\nabla}(\lambda).$$

The subgeneric situation

Suppose that Λ is subgeneric, i.e. $\widehat{\mathcal{W}}_\Lambda$ is isomorphic to the affine Weyl group of type A_1, and regular. In this case, each $\widehat{\mathcal{W}}_\Lambda$-orbit is a totally ordered set and we can define the successor bijection $\alpha \uparrow \cdot : \Lambda \to \Lambda$.

Theorem (with T. Arakawa)

- *Each $L(\lambda)$, $\lambda \in \Lambda$, admits a projective cover $\overline{P}(\lambda)$ in $\overline{\mathcal{O}}_\Lambda$. It fits into a short exact sequence*

$$0 \to \overline{\Delta}(\alpha \uparrow \lambda) \to \overline{P}(\lambda) \to \overline{\Delta}(\lambda) \to 0.$$

- *We have*

$$[\overline{\Delta}(\lambda) : L(\mu)] = \begin{cases} 1, & \text{if } \lambda \in \{\mu, \alpha \uparrow \mu\}, \\ 0, & \text{else.} \end{cases}$$

Restricted block decomposition

Let $\overline{\mathcal{O}} \subset \mathcal{O}$ be the full critical restricted subcategory and denote by

$$\overline{\mathcal{O}} = \prod_{\Gamma} \overline{\mathcal{O}}_{\Gamma}$$

the abstract block decomposition. Again, we identify Γ with the set $\{\gamma \in \widehat{\mathfrak{h}}^{\star} \mid L(\gamma) \in \overline{\mathcal{O}}_{\Gamma}\}$. For any $\mathcal{J} \subset \widehat{\mathfrak{h}}^{\star}$ that is *open*, i.e.

$$\mathcal{J} = \bigcup_{\gamma \in \mathcal{J}} \{\leq \gamma\},$$

we define $\overline{\mathcal{O}}_{\Gamma}^{\mathcal{J}} \subset \overline{\mathcal{O}}_{\Gamma}$ as the full subcategory of objects that have weights only in $\mathcal{J}$.

More results

Theorem (with T. Arakawa)

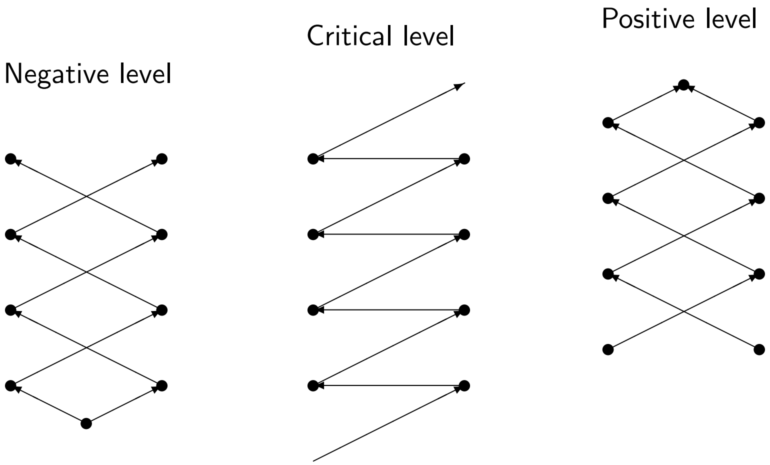
Let Γ *be a restricted index, let* $\mathcal{J}$ *be open and fix* $\gamma \in \Gamma \cap \mathcal{J}$.

- *The simple object* $L(\gamma)$ *has a projective cover* $\overline{P}^{\mathcal{J}}(\gamma)$ *in* $\overline{\mathcal{O}}_{\Gamma}^{\mathcal{J}}$.
- *Each projective in* $\overline{\mathcal{O}}_{\Gamma}^{\mathcal{J}}$ *has a restricted Verma flag and we have*

$$(\overline{P}^{\mathcal{J}}(\gamma) : \overline{\Delta}(\nu)) = [\overline{\Delta}(\nu) : L(\gamma)]$$

  for all $\nu \in \Gamma \cap \mathcal{J}$ *(Restricted BGG-reciprocity).*
- $[\overline{\Delta}(\lambda) : L(\nu)] \neq 0$ *implies* $\nu \in \widehat{\mathcal{W}}_{\Gamma}.\lambda$ *(Restricted Linkage principle).*

Restricted blocks are orbits



The Feigin–Frenkel conjecture

Each restricted critical index Γ is now a $\widehat{\mathcal{W}}_\Gamma$-orbit. As a partially ordered set it carries a variant of the generic Bruhat order.

- Hence, we expect the representation theory to be governed by the periodic module.

Conjecture

Suppose that Γ is sufficiently generic*. Let $\gamma \in \Gamma$ be dominant for the finite Weyl group action. For any $x, y \in \widehat{\mathcal{W}}_\Gamma$ we have*

$$[\overline{\Delta}(x.\gamma) : L(y.\gamma)] = p_{A_x, A_y}(1),$$

where $p_{-,-}$ denotes the periodic polynomial associated to $\widehat{\mathcal{W}}_\Gamma$.

Known for $\gamma = 0$ (Frenkel & Gaitsgory) and in the subgeneric cases (Arakawa & —).

Conjectural links