Given a $\mathbb{C}$x vty w/ a $\mathbb{C}$x VB $\begin{array}{c}\mathcal{E}\\ \downarrow\\ X\end{array}$ there is a <u>Chern Class</u> $c(\mathcal{E}) \in H^*(X)$

This is a <u>nonhomogeneous</u> cohom. class, and we write $c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \ldots$

where $c_i(\mathcal{E})$ is the homog. component in degree $2i$, and t is a dummy variable used to keep track of degrees. All degrees are even.

Chern classes satisfy:

1. $c_0(\mathcal{E}) = 1$
2. $c_k(\mathcal{E}) = 0$ if $\operatorname{rank}\mathcal{E} < k$
3. $c(\mathcal{E}) = 1$ if $\mathcal{E}$ is trivial
4. $0 \to \mathcal{E} \to \mathcal{B} \to \mathcal{F} \to 0$ $\qquad c(\overset{\mathcal{B}}{\mathcal{E}\oplus\mathcal{F}}) = c(\mathcal{E})c(\mathcal{F}) = 1 + (c_1(\mathcal{E}) + c_1(\mathcal{F}))t$
$+ (c_1(\mathcal{E})c_1(\mathcal{F}) + c_2(\mathcal{E}) + c_2(\mathcal{F}))t^2 + \ldots$
5. If
$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
so that $\mathcal{F} = f^*\mathcal{E}$ then $c(\mathcal{F}) = f^*c(\mathcal{E})$

<u>Exer 1:</u> $\mathcal{F}\ell = \mathcal{F}\ell(0,1,2,\ldots,n) = GL_n/B$ has n canonical line bundles $\mathcal{L}_i$, $i=1\ldots n$

Let $c(\mathcal{L}_i) \equiv 1 + x_i t$. What relations are imposed on x_i by

$\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots \oplus \mathcal{L}_n \cong \mathcal{O}^{\oplus n}$ trivial rank n bundle? (This splitting is really thought of in the compact version)

★[Supposing that these x_i generate $H^*(G/B)$ as an algebra, and these relations are all,]

show that $H^*(G/B) \cong \mathbb{C}[\mathfrak{h}] / \mathbb{C}[\mathfrak{h}]^W_+$

For variables $x_1,\ldots,x_n$ let $e_i(x_1,\ldots,x_n)$ denote the i^{th} <u>elementary</u> symmetric, $i=0,1,\ldots,n$
$h_i(x_1,\ldots,x_n)$ denote the i^{th} <u>complete</u> symmetric, $i \in \mathbb{N}$

<u>Exer 2:</u> In terms of the x_i, what is $c(\mathcal{L}_i \oplus \mathcal{L}_{i+1} \oplus \ldots \oplus \mathcal{L}_{i+k})$?

Exer 3: Let

$$\begin{array}{c} Fl(0,1,2,\dots,n) \\ \downarrow f \\ G/P_\mu = Fl_\mu = Fl(0,d_1,d_1+d_2,\dots,d_1+d_2+\dots+d_\ell) \end{array}$$

be the forgetful map.

$\Leftarrow$ Has canon. bundles $\mathcal{E}_1 \dots \mathcal{E}_\ell$ of rank $\mu = (d_1,\dots,d_\ell)$

What is $f^* c(\mathcal{E}_i)$?

Suppose, similar to $\star$ above, that $c_j(\mathcal{E}_i)$ generate $H^*(Fl_\mu)$, and that f^* is injective. Find a way to express $H^*(G/P)$ in terms of the x_i.

(Giveaway hint: $H^*(Gr(2,4)) \cong$ $\mathbb{C}[x_1+x_2, x_1x_2, x_3+x_4, x_3x_4] \big/ \left(\begin{array}{l} 0 = x_1+x_2+x_3+x_4 \\ = x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4 \\ = x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4 \\ = x_1x_2x_3x_4 \end{array} \right)$

deg = 1, 2, 1, 2)

Exer 4: Spose

$$(1+e_1(x_1\dots x_k)t + e_2(x_1,\dots x_k)t^2 + \dots + e_k(x_1,\dots x_k)t^k) \cdot (1+y_1t+y_2t^2+\dots) = 1$$

Show that $y_i = (-1)^i h_i(x_1,\dots,x_k)$

Exer 5: With similar assumptions to $\star$, what is $H^*(Fl(0,1,2,3,\dots,n,\infty))$?

[Hint: There is now an ∞-rank bundle $\mathcal{E}$. What is $c(\mathcal{E})$ w.r.t. $c(\mathcal{L}_i)$? Are there any relations between $c(\mathcal{L}_i)$?

What is $H^*Fl(0,1,2,3,\dots,n,N)$? It is naturally the quotient of $H^*Fl(0,1,\dots,n,\infty)$, by which chern classes? In what degree are the relations?

What is $H^*Gr(n,\infty)$? View it inside $H^*Fl(0,1,\dots,n,\infty)$.

Exer 6: Use this to give a different description of $H^*(Gr(k,n)) \cong \mathbb{C}[\underset{\deg=1}{e_1}, \underset{2}{e_2},\dots,\underset{k}{e_k}] \big/ \begin{array}{l} h_{n-k+1} \\ h_{n-k+2} \\ \vdots \end{array}$

What is $H^*(\mathbb{P}^n)$ explicitly? For fun, write h explicitly in terms of e for $k=2,3$. (Gross)

Rmk: For an understanding of $\star$, see an appendix of Hatcher "Alg Top" (look up Grassmanians in the index)