4. Exercises (Thursday)

Exercise 4.1. In the last exercise on Monday we gave an explicit realisation of a 2-dimensional (affine) Schubert variety X and constructed a resolution

$$\pi : \widetilde{X} \to X.$$

In this exercise we will examine the parity sheaves on X with the help of the resolution $\widetilde{X}$.

a) Recall that we have chosen a basis e_1, e_2 for V in order to define the action of a torus T on X. Consider the following filtration of $V_{-1} \oplus V_0$:

$$0 \oplus V_0 \subset \mathbb{C}e_2 \oplus V_0 \subset V_{-1} \oplus V_0.$$

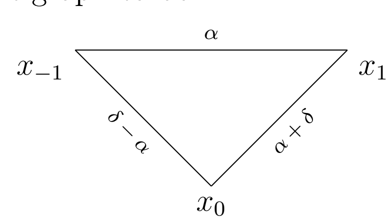
Show that this induces a filtration $Z_0 \subset Z_1 \subset Z_2$ of X with $Z_0 = \mathbb{A}0$ and $Z_i \setminus Z_{i-1} \cong \mathbb{A}^i$ for $i = 1, 2$.

b) Construct a similar stratification of $\widetilde{X}$ into affine spaces, so that π becomes a stratified even resolution.

c) Now let k be a field. From the above we know that $\pi_* k_{\widetilde{X}}[2]$ is a parity complex on X. Use the decomposition theorem and the fact that X is rationally smooth to decompose $\pi_* k_{\widetilde{X}}[2]$ if k is of characteristic zero.

d*) Let x_0 denote the unique singular point of X. Calculate the self-intersection of $\pi^{-1}(x_0) \subset \widetilde{X}$, and conclude that $\pi_* k_{\widetilde{X}}[2]$ splits as a direct sum of two (shifted) constant sheaves if $\operatorname{char} k \neq 2$. What happens if $\operatorname{char} k = 2$? (*Hint:* It might help to consider the restriction of π to $\pi^{-1}(U)$, where U is the open set consisting of those $W \in X$ which are transverse to $V_{-1} \oplus 0$.)

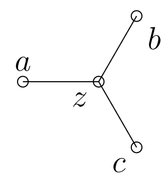
Exercise 4.2. *(In this exercise we see how the moment graph language can be used to obtain the result obtained topologically in Exercise 1.)* Let X be as in Exercise 4.1. Recall that in Exercise 1.8 on Monday we considered an action of a rank 3 torus $T \times \mathbb{C}^*$ on X and (hopefully!) calulated the moment graph to be



where α and δ are two characters of $T \times \mathbb{C}^*$.

Conduct the Braden-MacPherson algorithm with coefficients in $\mathbb{Z}_\ell$ to deduce that the parity sheaf $\mathcal{E}$ supported on X has a non-trivial stalk at x_0 if $\ell = 2$. Deduce that X is not 2-smooth.

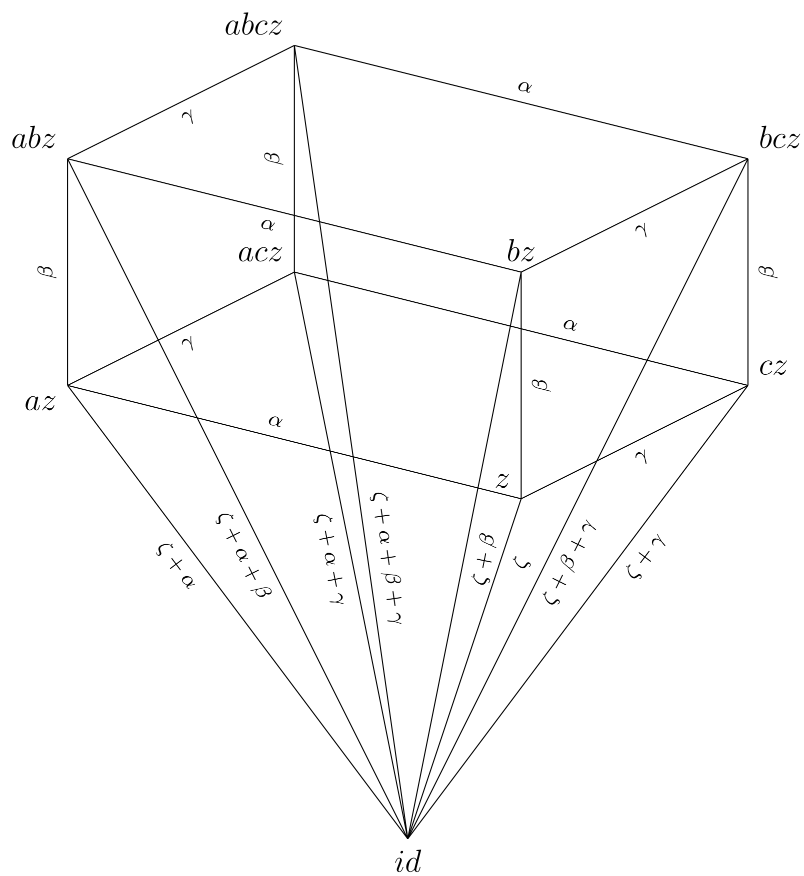
Exercise 4.3. Consider W of type D_4 with generators



and simple roots $\alpha, \beta, \gamma, \zeta$ such that $a = s_\alpha$, $b = s_\beta$, $c = s_\gamma$ and $z = s_\zeta$.

Consider the Schubert variety $X_{abcz} \subset G/P_{a,b,c}$.

a) Show that its moment graph is given by:



b) By performing the Braden-MacPherson algorithm over $\mathbb{Q}$, show that the Kazhdan-Lusztig polynomial $P_{abc,abczabc} = 1 + 2q$.

c) By performing the Braden-MacPherson alogorithm with coefficients in $\mathbb{Z}_2$, conclude that the parity sheaf corresponding supported on X is not isomorphic to the intersection cohomology complex. Is the parity sheaf perverse?

d*) Can you describe the singularity of X at the T-fixed point $abcz$ explicitly? (*Hint:* Use the Bott-Samelson resolution to identify X at $abcz$ with the contraction of the zero section of a line bundle on $(\mathbb{P}^1)^3$.) Hence describe the stalks of the intersection cohomology complex on X over $\mathbb{Z}$.

Exercise 4.4. (This is a hard one)

Let $\mathscr{F}$ be a sheaf on a moment graph. For a vertex x we define the costalk of $\mathscr{F}$ at x as

$$\mathscr{F}_x := \{m \in \mathscr{F}^x \mid \rho_{x,E}(m) = 0 \text{ for any edge } E \text{ adjacent to } x\}.$$

Let $\mathcal{G}$ be the Bruhat graph associated to a root system and $\mathcal{W}$ the Weyl group, $l\colon \mathcal{W} \to \mathbb{N}$ the length function. For $w \in \mathcal{W}$ denote by $\mathscr{B}(w)$ the Braden-MacPherson sheaf on $\mathcal{G}$ with parameter w, and by $\mathbf{C}(w) = \Gamma(\mathscr{B}(w))$ the global sections. Let k be a field and suppose that $(\mathcal{G}_{\leq w}, k)$ is a GKM-pair. One can show that $\mathbf{C}(w)$ is, up to a shift a self-dual module over the structure algebra $\mathcal{Z}$, i.e.

$$\operatorname{Hom}^\bullet(\mathbf{C}(w), S) \cong \mathbf{C}(w)[2l(w)].$$

- Show that this induces an isomorphism $\operatorname{Hom}^\bullet(\mathscr{B}(w)^x, S) \cong \mathscr{B}(w)_x[2l(w)]$ of graded S-modules.

Now let $\gamma\colon \mathfrak{h} \to \mathcal{D}$ be a generic character and consider the unital algebra homomorphism $S_k \to k[t]$ that is given by $H \mapsto \gamma(H)t$. For any S-module M we set $\overline{M} = M \otimes_{S_k} k$. Suppose that the quotient $X = \overline{\mathscr{B}(w)^x}/\overline{\mathscr{B}(w)_x}$ satisfies the Lefschetz property, i.e. suppose that

$$t^n\colon X_{[l(w)-n]} \to X_{[l(w)+n]}$$

is an isomorphism for any $n \geq 0$.

- Deduce the Kazdhan–Lusztig conjecture.