Today is Whiskey Day! We don't insist on elegantly completed, beautifully written and carefully texed solutions to all exercises.

3. Exercises (Wednesday)

Exercise 3.1. Let $\mathbb{Z}$-mod denote the category of finitely generated $\mathbb{Z}$-modules and $D^b(\mathbb{Z}\text{-mod})$ its bounded derived category. Consider the functor

$$\mathbb{D} = \mathrm{RHom}(-, \mathbb{Z}) : D^b(\mathbb{Z}\text{-mod})^{op} \to D^b(\mathbb{Z}\text{-mod})$$

and let $(D^{\leq 0}, D^{\geq 0})$ denote the standard t-structure on $D^b(\mathbb{Z}\text{-mod})$ with heart $\mathcal{C} = \mathbb{Z}$-mod.

a) Describe the dual t-structure, obtained by applying $\mathbb{D}$ to $(D^{\leq 0}, D^{\geq 0})$. Describe $\mathcal{C}' = \mathbb{D}(\mathcal{C}) \subset D^b(\mathbb{Z}\text{-mod})$ explicitly.
b) Show that if we regard $\mathbb{Z}$ as a complex concentrated in degree zero, then $\mathbb{Z}$ belongs to $\mathcal{C} \cap \mathcal{C}'$. Describe the kernel and cokernel of the map $\mathbb{Z} \to \mathbb{Z} : 1 \mapsto n$, first in $\mathcal{C}$, then in $\mathcal{C}'$.

Exercise 3.2. Let us go back to the setting of Exercise 2.3.

a) Calculate the stalks of $\mathbf{IC}(\mathbb{C}, \mathcal{L}_\lambda)$.

Now let $\mathcal{L}_A$ denote a rank n local system on $\mathbb{C}^*$ with monodromy given by $A \in GL_n(\mathbb{C})$.

b*) Calculate the stalks of $j_!\mathcal{L}_A$, $j_*\mathcal{L}_A$ and $j_{!*}\mathcal{L}_A$ at 0. Deduce that $j_{!*}$ is not an exact functor in general.

Exercise 3.3. *The intersection cohomology complex of the cone over a smooth projective variety.*

Any projective algebraic variety $Z \subset \mathbb{P}^n$, is given by the vanishing of certain homogeneous polynomials in $n+1$ variables. We obtain the *cone* X over Z as the zero set of these polynomials in $\mathbb{A}^{n+1}$. In this case $X \setminus \{0\}$ is a $\mathbb{C}^*$-bundle over Z, with Chern class equal to the pull-back of the Chern class of $\mathcal{O}(1)$ on $\mathbb{P}^n$.

Suppose that $Z \subset \mathbb{P}^n$ is smooth, and let X denote the cone over X. Calculate the stalk of $\mathbf{IC}(X)$ at 0 in terms of the action of $c_1(\mathcal{O}(1))$ on the cohomology of Z. (*Hint:* Use the Leray-Serre spectral sequence to relate the cohomology of X and Z.)

More generally, suppose that X is obtained by "contracting the zero section" of a vector bundle E on a smooth variety Z. This means that there is a morphism

$$\pi : E \to X$$

which is an isomorphism over $E \setminus Z$ (where we regard $Z \subset E$ as the zero section) and contracts Z to a point. Then X has a unique singular point x_0. Calculate the stalk of $\mathbf{IC}(X)$ at x_0.

Exercise 3.4. Let $\pi \colon \overline{X} \to X$ denote a surjective proper map. We say that π is *semi-small* if $\overline{X} \times_X \overline{X} \subset \overline{X} \times \overline{X}$ has dimension equal to $d_X = \dim X$. (Dimension always refers to complex dimension.)

(1) If π is semi-small, prove the inequality

$$\dim\{x \in X \mid \dim \pi^{-1}(x) \geq i\} \leq d_X - 2i$$

for all $i \geq 0$.

(2) Show that $\pi_* \underline{\mathbb{Q}}_{\overline{X}}[\dim X]$ is perverse. (Hint: Prove that a sheaf F is perverse if (and only if)

$$\dim \operatorname{supp} \mathcal{H}^{-i} F \leq i \text{ and } \dim \operatorname{supp} \mathcal{H}^{-i} \mathbb{D} F \leq i).$$

(3) Can one give a condition to guarantee that $\pi_* \underline{\mathbb{Q}}_{\overline{X}}[\dim X]$ is an intersection cohomology complex?

Exercise 3.5. Consider the map $\pi \colon \mathbb{R} \to \mathrm{pt}$. Show that $\pi_!$ does not have a right adjoint $\pi^! \colon Sh(\mathrm{pt}) \to Sh(\mathbb{R})$.