2. Exercises (Tuesday)

Exercise 2.1. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with standard Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Let $S = S(\mathfrak{h})$ and consider the corresponding deformed category $\mathcal{O}_S$. Let $M_S(\lambda)$ be the deformed Verma module with highest weight λ.

- Describe $M_S(\lambda)$ as explicitly as possible.
- Show that $\mathrm{Hom}_{\mathcal{O}_S}(M_S(\lambda), M_S(\mu)) = 0$ for $\lambda \neq \mu$.

Exercise 2.2. For $i = 0, 1, 2$ consider the ray

$$R_i := \{\lambda e^{2\pi i/3} \mid \lambda \in \mathbb{R}_{\geq 0}\} \subset \mathbb{C}.$$

Set $Z = R_1 \cup R_2 \cup R_3$ and let $i : Z \hookrightarrow \mathbb{C}$ denote the inclusion. Calculate $i^! \underline{\mathbb{Q}}_{\mathbb{C}}$, where $\underline{\mathbb{Q}}_{\mathbb{C}}$ denotes the constant sheaf on $\mathbb{C}$.

Exercise 2.3. Let $j : \mathbb{C}^* \to \mathbb{C}$ denote the open immersion. Compute the stalk $(j_* \underline{k}_{\mathbb{C}^*})_0$ and compare it with $H^*(j^{-1}(\{0\}, k)$. Same question for $(j_! \underline{k}_{\mathbb{C}^*})_0$.

Exercise 2.4. More generally, let $\mathcal{L}_\lambda$ denote a rank one local system on $\mathbb{C}^*$ with monodromy given by $\lambda \in \mathbb{C}^*$. Let $j : \mathbb{C}^* \hookrightarrow \mathbb{C}$ denote the inclusion. Describe the graded rank of the stalks of $j_! \mathcal{L}_\lambda$ and $j_* \mathcal{L}_\lambda$ at $0 \in \mathbb{C}$.

Exercise 2.5. Consider $\mathbb{R}^2$ with coordinates (x, y), let $\pi : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$ denote the first projection, and $i : \{0\} \hookrightarrow \mathbb{R}$ the inclusion. Consider the Cartesian diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{i'} & \mathbb{R}^2 \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \{0\} & \xrightarrow{i} & \mathbb{R} \end{array}$$

Let $H = \{(x, y) \mid xy = 1\} \subset \mathbb{R}^2$ denote a hyperbola in $\mathbb{R}^2$ and let $\mathbb{Q}_H$ denote the constant sheaf on H, extended by zero to $\mathbb{R}^2$.

a) Describe the sheaves $\pi_* \mathbb{Q}_H$ and $\pi_! \mathbb{Q}_H$.
b) Show that $i^! \pi_* \mathbb{Q}_H \cong \pi'_*(i')^! \mathbb{Q}_H$ but that $i^* \pi_* \mathbb{Q}_H \not\cong \pi'_*(i')^* \mathbb{Q}_H$.
c) If you did the above example without deriving your functors, redo it with derived functors! Why does nothing change in this case?

Exercise 2.6. Let $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$ denote the natural projection.

a) Show that π is a locally trivial fibration with fibre $\mathbb{C}^*$.

b) Is $\pi_* \underline{k}$ the direct sum of its cohomology sheaves? Hint: consider the "Leray-Serre spectral sequence" associated to this fibration.

Exercise 2.7. Let X be a variety endowed with a free action of a finite group H.

a) Convince yourself that $\mathrm{R}\Gamma(X, k)$ is a perfect complex of kH-modules (it can be represented by a bounded complex of projective kH-modules), and that $\mathrm{R}\Gamma(X/H, k)$ is obtained from that of X by taking derived fixed points.
b) Recover the cohomology of the real projective space.