There are far too many exercises. Complete as many (or as few) as you wish!

1. Exercises (Monday)

Exercise 1.1. Let $m \geq 2$ and consider the Coxeter group

$$W := \langle s, t \mid (st)^m = \mathrm{id} \rangle.$$

a) Describe the Bruhat order on W.
b) Show that for all $x \leq y$ in W, the Kazhdan-Lusztig polynomial $P_{x,y}$ is equal to 1.

Exercise 1.2. Let W denote the Weyl group S_4 with simple reflections s, t and u such that $su = us$.

a) Calculate the the set $\{x \in W \mid x \leq tsut\}$. How many elements are there of each length?
b) Calculate the Kazhdan-Lusztig polynomial $P_{t,tsut}$.
c*) Can you find other examples of non-trivial Kazhdan-Lusztig polynomials in $W = S_4$?

Exercise 1.3. Let $\mathfrak{g}$ be a finite dimensional, semisimple complex Lie algebra and fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Denote by $\mathfrak{h}^*$ the space of $\mathbb{C}$-linear forms on $\mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ be the Verma module and $L(\lambda)$ the simple module of highest weight λ, and denote by $P(\lambda)$ the projective cover of $L(\lambda)$ in $\mathcal{O}$.

Show that $\dim \mathrm{Hom}(P(\lambda), M) = [M : L(\lambda)]$ for all modules M of $\mathcal{O}$ and $\lambda \in \mathfrak{h}^*$.

Exercise 1.4. Let $\sim$ be the equivalence relation on $\mathfrak{h}^*$ that is generated by $\lambda \sim \mu$ if $[M(\lambda) : L(\mu)] \neq 0$. For any equivalence class $\Lambda \in \mathfrak{h}^*/\sim$ let $\mathcal{O}_\Lambda$ be the full subcategory of $\mathcal{O}$ that contains all modules M with $[M : L(\lambda)] \neq 0$ only if $\lambda \in \Lambda$.

Show that

$$\bigoplus_{\Lambda \in \widehat{\mathfrak{h}}^\star/\sim} \mathcal{O}_\Lambda \to \mathcal{O}$$

$$(M_\Lambda) \mapsto \bigoplus M_\Lambda$$

is an equivalence of categories. Hint: Use the *BGG-reciprocity:* $(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$.

Exercise 1.5. Let $\mathscr{F}$ be a sheaf on the moment graph $\mathcal{G}$. Denote by $\mathcal{V}$ its set of vertices. Show that the following are equivalent:

(1) The restriction map $\Gamma(\mathcal{V}, \mathscr{F}) \to \Gamma(U, \mathscr{F})$ is surjective for any open subset U of $\mathcal{V}$.

(2) The restriction map $\Gamma(\{\geqslant x\}, \mathscr{F}) \to \Gamma(\{> x\}, \mathscr{F})$ is surjective for any $x \in \mathcal{V}$.
(3) The map $\mathscr{F}^x \to \bigoplus_{E:\, x \to y} \mathscr{F}^E$ contains $\mathscr{F}^{\delta x}$ in its image.

Exercise 1.6. Let $V = \mathbb{C}^n$ equipped with the natural action of $T = (\mathbb{C}^*)^n$. Let $\mathrm{Gr}(k,n)$ denote the Grassmannian of k-planes in $\mathbb{C}^n$. Show that T has finitely many zero and one-dimensional orbits on $\mathrm{Gr}(k,n)$ and describe the moment graph of this action. (You might like to do the case $k = 1$, $Gr(k,n) = \mathbb{P}^{n-1}$ as a warm up!)

Exercise 1.7. Let $\langle \cdot | \cdot \rangle$ denote a symplectic form on a four dimensional vector space V and let $G = \mathrm{Sp}(V)$ denote the corresponding symplectic group.

a) Show that the variety X of isotropic 2-planes in V is a partial flag variety for G.
b) Fix an isotropic 2-plane L and consider
$$X_1 := \{L' \in X \mid \dim(L' \cap L) \geq 1\}.$$
Show that X_1 is a Schubert variety in X.
c) Show that X_1 is singular.
d*) Describe the singularity of X_1 explicitly as a quotient singularity and deduce that it is rationally smooth.

Exercise 1.8. Let V denote a two dimensional complex vector space, set $V_{-1} = V_0 = V$ and consider $V_{-1} \oplus V_0$ equipped with a nilpotent endomorphism
$$t : V_{-1} \oplus V_0 \to V_{-1} \oplus V_0 : (v_{-1}, v_0) \mapsto (0, v_{-1}).$$
Consider the variety
$$X = \{W \subset V_{-1} \oplus V_0 \mid \begin{matrix} W \text{ is a } t\text{-stable subspace,} \\ \dim W = 2 \end{matrix} \}$$

a) Show that X is naturally a closed subvariety of the Grassmannian of 2-planes in $V_{-1} \oplus V_0$, and is hence a projective variety.
b) Show that X is a projective variety and that an open neighbourhood of the point $W = 0 \oplus V_0 \in X$ is a singularity of type A_1 (i.e. the singularity given by the equation $x^2 = yz$ inside $\mathbb{A}^3$). (*Hint:* consider subspaces $W \subset V_{-1} \oplus V_0$ which are transversal to V_{-1} as graphs of linear maps $V_0 \to V_{-1}$. What is the condition to be t-stable?)
c) We now define a torus action on X. Let us assume $V \cong \mathbb{C}^2$ with the standard action of a rank two torus T. Let $\mathbb{C}^*$ act on $V_{-1} \otimes V$ by $\lambda \cdot (v_{-1}, v_0) = (\lambda v_{-1}, v_0)$. Show that we get an induced action of $T \times \mathbb{C}^*$ on X. Show that $T \times \mathbb{C}^*$ has finitely

many zero and one dimensional orbits on X and compute the moment graph.

d) Consider the space

$$\widetilde{X} = \left\{ W_1 \subset W \subset V_{-1} \oplus V_0 \mid \begin{array}{c} W_1, W \text{ } t\text{-stable subspaces,} \\ \dim W_1 = 1, \dim W = 2 \end{array} \right\}$$

Show that $\widetilde{X}$ is a smooth projective variety and that the map

$$\pi : \widetilde{X} \to X : (W_1, W) \mapsto W$$

is a resolution of singularities. Show that $T \times \mathbb{C}^*$ also acts naturally on $\widetilde{X}$ and describe the map $\widetilde{X} \to W$ in terms of moment graphs.

(In fact, X is an example of a Schubert variety in the "affine Grassmannian" for GL_2 and π is its Bott-Samelson resolution.)

Exercise 1.9. Compute the cohomology with integer coefficients of a sphere, of a compact surface of genus g, of a real or complex projective space (particularly $\mathbb{RP}^3$).

Exercise 1.10. [1] Let $\dot{S}$ denote a sphere minus three points, and $\dot{T}$ denote a torus minus one point. Show that $\dot{S}$ and $\dot{T}$ are homotopic, but not homeomorphic.

Hints (two methods; you can try to find your own):

(1) try to remove something; [2]

(2) consider the morphism from H^1_c to H^1 and use Hodge structures.

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[1] This exercise is not very relevant, it is there just for fun. We thank Luca Migliorini for sharing it with us.

[2] Model: see that $\mathbb{R}$ and $\mathbb{R}^2$ are not hemeomorphic by removing one point.