Hypertoric category $\mathcal{O}$

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Sheaves in Representation Theory, Skye
May 28, 2010

Let $(\mathfrak{M}, \omega)$ be a smooth variety over $\mathbb{C}$ with an algebraic symplectic form. Assume that

- The map $\mathfrak{M} \to \mathfrak{M}_0 := \operatorname{Spec} \Gamma(\mathfrak{M}, \mathcal{O}_{\mathfrak{M}})$ is a resolution of singularities.
- There is an action of $\mathbb{S} = \mathbb{C}^*$ on $\mathfrak{M}$ acting on ω and on nonconstant functions with positive weight. (contracts $\mathfrak{M}_0$ to 0)

ω induces a Poisson structure $\{\cdot, \cdot\}$ on $\mathbb{C}[\mathfrak{M}_0]$. A quantization of $\mathfrak{M}_0$ is an $\mathbb{N}$-filtered algebra U equipped with an isomorphism $\operatorname{gr} U \cong \mathbb{C}[\mathfrak{M}_0]$ so that

$$\text{if } \; x \in F_k U, y \in F_\ell U, \text{ then } [x, y] = \{f, g\} \mod F_{k+\ell-2} U.$$

There will in general be many different quantizations U_λ, parametrized by $\lambda \in H^2(\mathfrak{M}; \mathbb{C})$.

Examples satisfying these hypotheses include many varieties and algebras of interest in representation theory:

- $\mathfrak{M} = T^*(G/B)$, $\mathfrak{M}_0 =$ the nilcone;
  $U_\chi = U(\mathfrak{g})/I_\chi$, $\chi \colon Z(U(\mathfrak{g})) \to \mathbb{C}$.

- $\mathfrak{M} = \widetilde{\mathbb{C}^2/\Gamma}$, a crepant resolution of $\mathbb{C}^2/\Gamma$, for $\Gamma \subset SL_2(\mathbb{C})$ finite, or more generally the Hilbert scheme $\mathfrak{M} = \mathrm{Hilb}_n(\widetilde{\mathbb{C}^2/\Gamma})$;
  $U_\lambda =$ a spherical rational Cherednik algebra of $S_n \wr \Gamma$

- $\mathfrak{M}$ is a quiver variety, à la Nakajima;
  $U =?$.

- $\mathfrak{M}$ is a hypertoric variety;
  U is an algebra we call the hypertoric enveloping algebra.

Hypertoric varieties

Standard action of $T^n = (\mathbb{C}^*)^n$ on $\mathbb{C}^n$ induces action on $T^*\mathbb{C}^n$.
Take a connected subtorus $K \subset T^n$. For a character $\lambda \colon K \to \mathbb{C}^*$, define

$$\mathfrak{M}_\lambda = (T^*\mathbb{C}^n)/\!\!/\!\!/\!\!/_\lambda K := \mu^{-1}(0)/\!\!/_\lambda K,$$

where $\mu \colon T^*\mathbb{C}^n \to \mathfrak{k}^*$ is the moment map of the K-action:

$$\begin{aligned} T^*\mathbb{C}^n &\longrightarrow (\mathfrak{t}^n)^* \longrightarrow \mathfrak{k}^* \\ (z_i, w_i) &\mapsto (z_i w_i) \end{aligned}$$

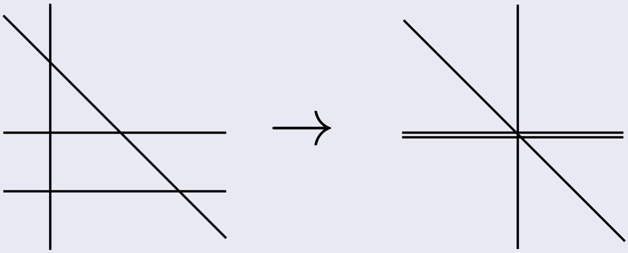
For general λ, $\mathfrak{M}_\lambda$ is ($\mathbb{Q}$-)smooth and symplectic; at the other extreme,

$$\mathfrak{M}_0 = \mathrm{Spec}\left(\mathbb{C}[z_i, w_i]^K / \mu^*(\mathfrak{k})\right)$$

is affine and highly singular. There is a natural resolution $\mathfrak{M}_\lambda \to \mathfrak{M}_0$.

The geometry of $\mathfrak{M}_\lambda$ is governed by the hyperplane arrangement $\mathcal{H}_\lambda := (V_\lambda, \{H_1, \ldots, H_n\})$, where

$$V_\lambda = \text{inverse image of } \lambda \text{ under } (\mathfrak{t}^n_{\mathbb{R}})^* \to \mathfrak{k}^*_{\mathbb{R}}, \quad H_i = V_\lambda \cap \{x_i = 0\}.$$



$$\mathfrak{M}_\lambda \longrightarrow \mathfrak{M}_0$$

Each chamber gives a Lagrangian toric subvariety: set $z_i = 0$ or $w_i = 0$ depending on which side of H_i you're on.

Quantizing the hypertoric variety

The Weyl algebra $\mathbb{D} = \mathbb{D}(\mathbb{C}^n)$ quantizes $T^*\mathbb{C}^n$. Define

$$U := \mathbb{D}^K \quad \text{“enveloping algebra”}$$
$$H := \mathbb{D}^{T^n} \quad \text{“Cartan subalgebra”}$$

H is a polynomial algebra over $h_i^+ = z_i\partial_i$ (or $h_i^- = \partial_i z_i = h_i^+ + 1$). Define $\mu_q \colon \mathfrak{k} \to H$ by

$$(a_1, \ldots, a_n) \mapsto \sum a_i h_i^+.$$

Its image generates the center $Z(U)$, and

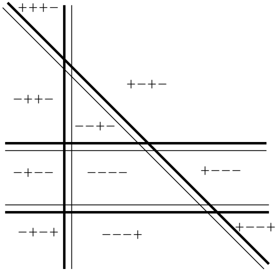
$$U_\lambda := U/U\langle(\mu_q - \lambda)(\mathfrak{k})\rangle$$

is a quantization of $\mathfrak{M}_0$.

Weight modules

A weight module is fin. gen. U_λ-module on which H acts locally finitely. (assume integral weights: i.e., in $\Lambda = \{v \in V_\lambda \mid \text{ all } h_i^+(v) \in \mathbb{Z}\}$)

Musson and van den Bergh computed category of weight modules.



Simples L_α are indexed by "feasible chambers" $\alpha \in \mathcal{F}$: regions given by $h_i^+ \geq 0$ or $h_i^- \leq 0$ which contain a lattice point.

All such lattice points have one-dimensional weight spaces.

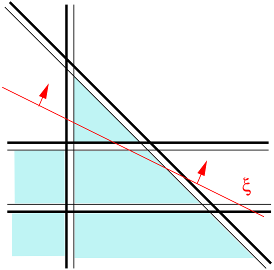
Say that λ is "generic" if $\mathcal{H}_\lambda$ is simple (no point is on more than dim V_λ hyperplanes) and $\mathcal{F} =$ set of regions of $\mathcal{H}_\lambda$.

category $\mathcal{O}$

Take $\xi \colon \mathbb{C}^* \to T^n/K$; gives linear function on affine space V_λ.
Assume ξ *generic*: not constant on any 1-flat of arrangement.
Let $U_\lambda^+ \subset \mathbb{D}$ be span of elements of nonneg ξ-weight.

Definition

$\mathcal{O}(K, \lambda, \xi)$ = the category of (integral) weight modules for U_λ which are U_λ^+-locally finite.



Simples are L_α for α in subset $\mathcal{B} \subset \mathcal{F}$ of ξ-bounded chambers.

Lagrangian toric varieties for $\alpha \in \mathcal{B}$ are closures of cells $C_\alpha := \{q \in \mathfrak{M} \mid \lim_{t\to 0} \xi(t) \cdot q = p_\alpha\}$, $p_\alpha \in \mathfrak{M}^{T^n/K}$ is the fixed point corresponding to the ξ-maximal point of the chamber.

The hypertoric category $\mathcal{O}(K, \lambda, \xi)$ is more correctly analogous to a variant $\mathcal{O}'$ of category $\mathcal{O}$

Category $\mathcal{O}(\mathfrak{g})$	Category $\mathcal{O}'(\mathfrak{g})$
$\mathfrak{h}$ acts semisimply	$\mathfrak{h}$ acts locally finitely
$Z(U(\mathfrak{g}))$ acts locally finitely	$Z(U(\mathfrak{g}))$ acts semisimply

But Soergel showed that $\mathcal{O}'(\mathfrak{g})_\chi$ is equivalent to $\mathcal{O}(\mathfrak{g})_\chi$ for χ integral regular.

Quiver description

Theorem

$\mathcal{O}(K, \lambda, \xi)$ is equivalent to modules over an algebra $A = A(K, \lambda, \xi)$.

$Q_1 :=$ the path algebra of quiver

$$- \rightleftarrows +$$

$Q_n := Q_1 \otimes_{\mathbb{C}} Q_2 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} Q_1$.

$$A := (e_{\mathcal{F}}\, Q_n\, e_{\mathcal{F}}) / \langle e_{\mathcal{F} \setminus \mathcal{B}}, \vartheta(\mathfrak{k}) \rangle,$$

where $\vartheta \colon \mathfrak{t}^n = \mathbb{C}^n \to Q_n$ sends i^{th} basis vector to sum of all loops flipping the sign in the i^{th} place.

Assume λ, ξ are generic.

Theorem

$\mathcal{O}(V, \lambda, \xi)$ is highest weight (A is quasi-hereditary): there are

- *a partial order on set $\mathcal{B}$ of idempotents,*
- *standard and projective objects M_α, P_α, $\alpha \in \mathcal{B}$, and*
- *surjections $P_\alpha \to M_\alpha \to L_\alpha$ so that*

$\ker(M_\alpha \to L_\alpha)$ has filtration with subquotients L_β, $\beta < \alpha$, and $\ker(P_\alpha \to M_\alpha)$ has filtration with subquotients M_γ, $\gamma > \alpha$.

Partial order: $\alpha \leq \beta$ iff there is a path $v_\alpha \to v_\beta$ between highest vertices of the chambers, running along edges in $\mathcal{H}_\lambda$ in direction of increasing ξ.

Multiplicities in standards: $[M_\beta : L_\alpha] = 1$ if chamber α lies in downward cone from v_β, 0 otherwise. (K-L polynomials are all 0 or 1!)

Note that A is graded, although it's not obvious from definition of $\mathcal{O}$.

Theorem

The algebra A is Koszul: in category of graded A-modules,

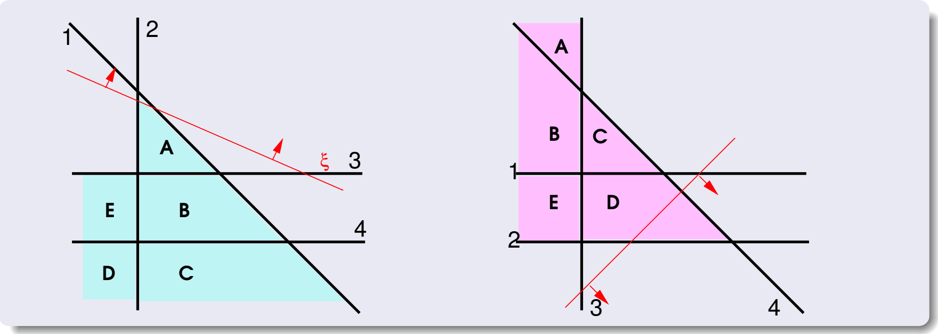
$$\text{for any } \alpha, \beta, \quad \mathrm{Ext}^i(L_\alpha, L_\beta)_j = 0 \text{ if } i \neq j.$$

The Koszul dual ring $A^! := \mathrm{Ext}^*_A(A_0, A_0)$ is isomorphic to $A(K^!, -\xi, -\lambda)$, where $\mathfrak{k}^!$ is the orthogonal complement to $\mathfrak{k} \subset \mathfrak{t}^n = \mathbb{C}^n$ under the standard inner product.

Gives a derived equivalence $D^b(A - gr) \cong D^b(A^! - gr)$ sending simples to projectives, injectives to simples, and standards to standards (with shifts).

The correspondence $(K, \lambda, \xi) \leftrightarrow (K^!, -\xi, -\lambda)$ is called Gale duality.

In particular, there is an order-reversing bijection $\mathcal{B} \leftrightarrow \mathcal{B}^!$ between the sets of bounded feasible chambers (in fact $\mathcal{B} = \mathcal{B}^!$ as sets of sign vectors).



The relation "share a codimension one wall" (= has a nonzero Ext^1) is also preserved by Gale duality.

We can also see the algebra A on the dual side:

$\mathfrak{M}^!$	hypertoric variety defined by $(K^!, -\xi, -\lambda)$.
$X^!_\alpha, \alpha \in \mathcal{B}$	Lagrangian toric subvarieties for bounded chambers.

Then

$$B(K^!, -\xi, -\lambda) := \bigoplus_{\alpha,\beta\in\mathcal{B}} H^*(X_\alpha \cap X_\beta)$$

(with a strange grading) has an associative convolution product, making it isomorphic to $A(V, \lambda, \xi)$.

This is analogous to Stroppel and Webster's description of the algebra governing parabolic category $\mathcal{O}$ for a maximal parabolic in $\mathfrak{gl}_n$ via a convolution product on intersections of components of a Springer fiber.

The B-ring description of $\mathcal{O}$ makes it easy to see the center:

Theorem

The map

$$H^*(\mathfrak{M}^!) \mapsto \bigoplus_{\alpha \in \mathcal{B}} H^*(X_\alpha \cap X_\alpha) \mapsto \bigoplus_{\alpha,\beta \in \mathcal{B}} H^*(X_\alpha \cap X_\beta)$$

gives an isomorphism

$$H^*(\mathfrak{M}^!) \cong Z(B(K^!, -\xi, -\lambda)) = Z(A(K, \lambda, \xi)).$$

Localization

When λ and ξ are generic, the category $\mathcal{O}(K, \lambda, \xi)$ localizes to modules over a sheaf of rings on $\mathfrak{M}_\lambda$, quantizing the sheaf of regular functions.

To quantize as a sheaf, need to introduce a parameter: work over $\mathbb{C}((\hbar))$, and instead of $[\partial_i, z_i] = 1$, use $[w_i, z_i] = \hbar$.

Quantum hamiltonian reduction has a sheaf-theoretic version, giving sheaf $\mathcal{U}_\lambda$ of $\mathbb{C}((\hbar))$-modules on $\mathfrak{M}_\lambda$, whose $\mathbb{S}$-invariant sections $\Gamma_{\mathbb{S}}(\mathcal{U}_\lambda) \cong U_\lambda$.

Then $\mathcal{O}(K, \lambda, \xi)$ is equivalent to a certain subcategory of $\mathbb{S}$-equivariant $\mathcal{U}_\lambda$-modules which are set-theoretically supported on

$$\mathfrak{M}^+ = \bigcup_{\alpha \in \mathcal{B}} X_\alpha = \{p \in \mathfrak{M} \mid \lim_{t \to 0} \xi(t) \cdot p \text{ exists.}\}.$$

This gives a cycle map $K(\mathcal{O}) \to H_c^{\dim_{\mathbb{C}} \mathfrak{M}}(\mathfrak{M}^+) \cong \bigoplus_{\alpha \in \mathcal{B}} \mathbb{Z}$.

Left cells

For any $\alpha \in \mathcal{F}$, let Δ_α be its chamber in the arrangement $\mathcal{H}_\lambda$, and let $\Delta_{\alpha,0}$ be its "limit" in the central arrangement $\mathcal{H}_0$.

Theorem

For any $\alpha, \beta \in \mathcal{F}$, we have

$$\mathrm{Ann}_{U_\lambda} L_\alpha = \mathrm{Ann}_{U_\lambda} L_\beta$$

if and only if $\mathbb{R}\Delta_{\alpha,0} = \mathbb{R}\Delta_{\beta,0}$ *and* $\Delta_\alpha/\mathbb{R}\Delta_{\alpha,0} = \Delta_\beta/\mathbb{R}\Delta_{\beta,0}$.

The equivalence classes are called left cells.

Right cells

For any two generic $\lambda, \lambda' \in \mathfrak{k}^*$, there is a translation functor

$$T_\lambda^{\lambda'} : \mathcal{O}(K, \lambda, \xi) \to \mathcal{O}(K, \lambda', \xi).$$

Theorem

The following are equivalent for any α and β giving ξ-bounded regions (possibly for different λ!)

- $L_\beta^{\lambda'}$ *appears as a subquotient of* $T_\lambda^{\lambda'} L_\alpha^\lambda$ *for some* λ, λ'
- $\mathrm{Supp}(\mathrm{gr}\, L_\beta^{\lambda'}) \subset \mathrm{Supp}(\mathrm{gr}\, L_\alpha^\lambda) \subset \mathfrak{M}_0$
- $\Delta_{\alpha,0} \subset \Delta_{\beta,0}$.

The right cells are the equivalence classes generated by this relation.

Two-sided cells are the smallest subsets of $\mathcal{B}$ which are unions of left and right cells. α, β are in the same two-sided cell iff $\mathbb{R}\Delta_{\alpha,0} = \mathbb{R}\Delta_{\beta,0}$.

The sets $\mathbb{R}\Delta_{\alpha,0}$ are certain flats of the central arrangement $\mathcal{H}_0$, called relevant flats. They index strata in the coarsest possible stratification of $\mathfrak{M}_0$, by Poisson leaves.

Theorem

The bijection between simples of $\mathcal{O}(K, \lambda, \xi)$ and $\mathcal{O}(K^!, -\xi, -\lambda)$ induced by Koszul duality interchanges left and right cells, and the bijection on two-sided cells induces an order-reversing bijection between strata of $\mathfrak{M}_0$ and $\mathfrak{M}_0^!$.

Derived equivalences

The categories $\mathcal{O}(K, \lambda, \xi)$ for different choices of generic λ and ξ are not equivalent, but they are derived equivalent. For varying λ, the equivalences are translation functors; for varying ξ, there are shuffling functors, which are Koszul dual to translation functors.

The categories are not *canonically* equivalent, however. $T_{\lambda}^{\lambda'} T_{\lambda'}^{\lambda}$ is not the identity functor!

The translation functors generate an action of $\pi_1(M_{\mathbb{C}})$ on $D^b(\mathcal{O}(K, \lambda, \xi))$, where $M_{\mathbb{C}}$ is the complement of the complexification of the secondary arrangement: the arrangement in $\mathfrak{k}_{\mathbb{R}}^*$ whose walls give the non-generic λ.