9. Exercise Sheet "Lie algebras and Chevalley groups"

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Exercise 1. (Schriftlich) Let $L = \mathfrak{sl}_n(\mathbb{C})$ and $H \subseteq L$ be the usual abelian subalgebra of diagonal matrices. Let $\{h_i, e_i, f_i \mid 1 \leq i \leq n-1\}$ be as in Example 2.2.7; also recall that

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n, i \neq j\}, \qquad L_{\varepsilon_i - \varepsilon_j} = \langle e_{ij} \rangle_{\mathbb{C}}.$$

We set $\mathbf{e}_\alpha^+ := (-1)^j e_{ij}$ for $\alpha = \varepsilon_i - \varepsilon_j$, $i \neq j$. Show that the collection $\{\mathbf{e}_\alpha^+ \mid \alpha \in \Phi\}$ satisfies the conditions in Corollary 2.7.11. In particular, we have $\mathbf{e}_{\alpha_i}^+ = -(-1)^i e_i$ and $\mathbf{e}_{-\alpha_i}^+ = (-1)^i f_i$ for $1 \leq i \leq n-1$; furthermore, $h_i^+ = [e_i, \mathbf{e}_{-\alpha_i}^+] = (-1)^i h_i$.

[*Hint*. The highest root is $\alpha_0 = \alpha_1 + \ldots + \alpha_{n-1} = \varepsilon_1 - \varepsilon_n$; choose $\mathbf{e}_{\alpha_0} := (-1)^n e_{1n}$ and then follow the inductive procedure in Definintion 2.7.6 to define $\mathbf{e}_\alpha$ for all $\alpha \in \Phi^+$. Taking $(-1)^n e_{1n}$ instead of just e_{1n} yields the normalisation in Remark 2.7.7.]

Exercise 2. (Schriftlich) As in Exercise Sheet 7, let $L = \mathfrak{go}_4(Q_4, \mathbb{C})$, where

$$Q_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad Q_4^{\mathrm{tr}} = -Q_4.$$

Starting with the Chevalley generators specified on Exercise Sheet 7, determine a collection of elements $\{\mathbf{e}_\alpha^+ \mid \alpha \in \Phi\} \subseteq L$ as in Corollary 2.7.11.

Abgabe: bis Donnerstag, 25.6., 11:00 Uhr.