8. Exercise Sheet "Lie algebras and Chevalley groups"

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Back to the general setting: L is a finite-dimensional Lie algebra over $\mathbb{C}$ and $H \subseteq L$ is an abelian subalgebra such that (L, H) is of Cartan–Kiliing type with repsect to $\Delta = \{\alpha_i \mid i \in I\}$. Let $\Phi = \Phi^+ \cup \Phi^-$ be the set of roots of L.

Exercise 1. (Schriftlich) Let $i \in I$ and $\beta \in \Phi$ such that $\beta \neq \pm\alpha_i$. Consider the α_i-string through β:

$$\beta - q\alpha_i, \quad \ldots, \quad \beta - \alpha_i, \quad \beta, \quad \beta + \alpha_i, \quad \ldots, \quad \beta + p\alpha_i.$$

Let $p' := \max\{n \geq 0 \mid \beta + n\alpha_i \in \Phi\}$; note that $p' \geq p$. Consider the subspace

$$M' := M \oplus L_{\beta+(p+1)\alpha_i} \oplus \ldots \oplus L_{\beta+p'\alpha_i} \subseteq L, \qquad \text{with } M \text{ as in Remark 2.2.10.}$$

(Here, it could happen that $L_{\beta+m\alpha_i} = \{0\}$ for some $p + 1 \leq m < p'$.) Show that M' also is an S_i-submodule of L. By looking again at the eigenvalues of h_i, deduce that $p = p'$. Similarly, show that $q = \max\{n \geq 0 \mid \beta - n\alpha_i \in \Phi\}$.

Exercise 2. (Schriftlich) Consider the basis $\mathbf{B} = \{h_j^+ \mid j \in I\} \cup \{\mathbf{e}_\alpha^+ \mid \alpha \in \Phi\}$ of L in Remark 2.7.4. Define a linear map $\tilde{\omega}\colon L \to L$ by

$$\tilde{\omega}(h_j^+) := h_j^+ \quad (j \in I) \qquad \text{and} \qquad \tilde{\omega}(\mathbf{e}_\alpha^+) := \mathbf{e}_{-\alpha}^+ \quad (\alpha \in \Phi).$$

Using (L1), (L2), (L3) in Theorem 2.72 (or the formulas in Remark 2.7.4) verify that

$$\tilde{\omega} \circ \mathrm{ad}_L(e_i) = \mathrm{ad}_L(f_i) \circ \tilde{\omega} \qquad \text{and} \qquad \tilde{\omega} \circ \mathrm{ad}_L(h_i) = -\mathrm{ad}_L(h_i) \circ \tilde{\omega}$$

for all $i \in I$. Use Exercise 1.1.8(c) of the lecture notes to deduce that $\omega := -\tilde{\omega}\colon L \to L$ is a Lie algebra automorphism such that

$$\omega(e_i) = f_i, \qquad \omega(f_i) = e_i, \qquad \omega(h_i) = -h_i \qquad (i \in I).$$

This is called the *Chevalley involution* of L; we have $\omega^2 = \mathrm{id}_L$.

Exercise 3. Verify the entries in Table 3 (p. 87) of the lecture notes.

Abgabe: bis Donnerstag, 18.6., 11:00 Uhr.