7. Exercise Sheet "Lie algebras and Chevalley groups"

Professor Meinolf Geck, SoSe 2020

Exercise 1. (Schriftlich) The purpose of this exercise is to work out the details in Section 2.5 (about classical Lie algebras) for one specific example. Let $L = \mathfrak{go}_4(Q_4, \mathbb{C})$, where

$$Q_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad Q_4^{\text{tr}} = -Q_4.$$

A basis of L is given by Proposition 1.6.6(b); we have $\dim L = 10$. The diagonal matrices $H \subseteq L$ are described by Remark 1.6.7.

(a) Determine the set $N^+ \subseteq L$ of all strictly upper triangular matrices in L; similarly for N^-, the set of all strictly lower triangular matrices in L. (Both N^+ and N^- should be subspaces of L of dimension 4.)

(b) Verify the details in Lemma 2.5.3 and Proposition 2.5.4 for L. In particular, justify that $\Phi = \{\pm\varepsilon_1 \pm \varepsilon_2, \pm 2\varepsilon_1, \pm 2\varepsilon_2\}$ in this case. For each $\alpha \in \Phi$, determine a basis element (a matrix in L) of the weight space L_α.

(c) With the notation in Remark 2.5.5 and Corollary 2.5.6 we also have

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + 2\alpha_2)\}.$$

Write down explicitly α_1 and α_2 as a linear combination of ε_1 and ε_2.

(d) In Proposition 2.5.8 it is shown that L is of Cartan–Killing type with respect to H and $\Delta = \{\alpha_1, \alpha_2\}$. Show that the following elements are Chevalley generators for L:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad f_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad h_1 = [e_1, f_1] = \operatorname{diag}(0, 1, -1, 0);$$

$$e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad h_2 = [e_2, f_2] = \operatorname{diag}(1, -1, 1, -1).$$

Check the relations $[h_1, e_2] = -e_2$ and $[h_2, e_1] = -2e_1$; determine the structure matrix A of L.

(e) For each $\alpha \in \Phi$, express $h_\alpha \in H$ as a linear combination of h_1 and h_2.

Abgabe: bis Mittwoch, 10.6., 11:00 Uhr.