6. Exercise Sheet "Lie algebras and Chevalley groups"

Professor Meinolf Geck, SoSe 2020

Exercise 1.
Show that the following matrices can not be the structure matrix of a (finite-dimensional) Lie algebra of Cartan–Killing type.

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & a & b \\ a' & 2 & -1 \\ b' & -3 & 2 \end{pmatrix}$$

where $a, a', b, b' \in \mathbb{Z}_{\leq 0}$ are such that $aa' \geq 1$ (and $b = b' = 0$) or $bb' \geq 1$ (and $a = a' = 0$) or $aa' \geq 1$ and $bb' \geq 1$.

Exercise 2. (Schriftlich) For $m \geq 2$ we define a matrix $C_m \in M_m(\mathbb{Z})$ by

$$C_m := \begin{pmatrix} 2 & -1 & & & & \\ -2 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix},$$

where all non-specified entries are 0. We will see later that this is the structure matrix of the Lie algebra $L = \mathfrak{go}_{2m}(Q_{2m}, \mathbb{C})$, where $Q_{2m}^{\mathrm{tr}} = -Q_{2m}$.

Apply the `Python` function `rootsystem` (defined on p. 61 of the lecture notes) to $A = C_n$, for small values $m = 2, 3, 4, 5, 6, \ldots$ (Of course, you can also program that function in any other programming language of your choice.) Try to guess a general description of the set $\mathscr{C}^+(C_m)$ from these computations; what should $|\mathscr{C}^+(C_m)|$ be? Can you prove your guesses?

Exercise 3. (Schriftlich) We consider the Weyl group $W = \langle s_i \mid i \in I \rangle \subseteq \mathrm{GL}(H^*)$, as in Section 2.3. Let $w \in W$. We define the "length" of w, denoted $\ell(w)$, as follows. We set $\ell(\mathrm{id}) = 0$. Now let $w \in W$, $w \neq \mathrm{id}$. Then

$$\ell(w) := \min\{r \geq 1 \mid \exists\, i_1, \ldots, i_r \in I \text{ such that } w = s_{i_1} \cdots s_{i_r}\}.$$

In particular, $\ell(s_i) = 1$ for all $i \in I$.

(a) Show that $\ell(w) = \ell(w^{-1})$ for all $w \in W$.

(b) Let $w \in W$ and $i \in I$. Show that $\ell(ws_i) = \ell(w) \pm 1$.

(c) If $w = \mathrm{id}$, then we clearly have $\ell(ws_i) > \ell(w)$ for all $i \in I$. Show that the converse also holds, that is, if $w \in W$ is such that $\ell(ws_i) > \ell(w)$ for all $i \in I$, then $w = \mathrm{id}$.

Abgabe: bis Donnerstag, 28.5., 11:00 Uhr.