4. Exercise Sheet "Lie algebras and Chevalley groups"

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Exercise 1. Let V be a vector space over k and $\beta\colon V \times V \to k$ be a bilinear map. Define $\mathfrak{go}(V, \beta)$ to be the set of all $\varphi \in \mathrm{End}(V)$ such that $\beta(\varphi(v), w) + \beta(v, \varphi(w)) = 0$ for all $v, w \in V$. (The symbol "$\mathfrak{go}$" stands for "general orthogonal".) Show that $\mathfrak{go}(V, \beta)$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

Exercise 2. Now assume that $n = \dim V < \infty$ and let $B = \{v_1, \ldots, v_n\}$ be a basis of V. We form the corresponding Gram matrix $Q = \big(\beta(v_i, v_j)\big)_{1 \leq i,j \leq n} \in M_n(k)$. Let $\varphi \in \mathrm{End}(V)$ and $A = (a_{ij}) \in M_n(k)$ be the matrix of φ with respect to B. Then show that $\varphi \in \mathfrak{go}(V, \beta) \Leftrightarrow A^{\mathrm{tr}}Q + QA = 0$, where A^{tr} denotes the transpose matrix. Hence, $\mathfrak{go}(V, \beta)$ is isomorphic to the Lie subalgebra

$$\mathfrak{go}_n(Q, k) := \{A \in M_n(k) \mid A^{\mathrm{tr}}Q + QA = 0\} \subseteq \mathfrak{gl}_n(k).$$

Exercise 3. (Schriftlich) In the setting of Exercice 2, assume that

$$Q = Q_n := \begin{bmatrix} 0 & \cdots & 0 & \delta_n \\ \vdots & \iddots & \iddots & 0 \\ 0 & \delta_2 & \iddots & \vdots \\ \delta_1 & 0 & \cdots & 0 \end{bmatrix} \in M_n(k) \qquad (n \geq 2)$$

where $\delta_i = \pm 1$ are signs such that $Q_n^{\mathrm{tr}} = \pm Q_n$; note that $\det(Q_n) = \pm 1$ and so Q_n is invertible.

(a) Show that $Q_n^{-1} = Q_n^{\mathrm{tr}}$ and that $A \in \mathfrak{go}_n(Q_n, k) \Rightarrow A^{\mathrm{tr}} \in \mathfrak{go}_n(Q_n, k)$.

(b) Determine the diagonal matrices in $\mathfrak{go}_n(Q_n, k)$ for all $n \geq 2$.

(c) Let $n = 2$. Explicitly determine $\mathfrak{go}_2(Q_n, k)$. (Distinguish the cases where $\delta_1 = \delta_2$ and $\delta_1 = -\delta_2$.)

(d) Assume that $\mathrm{char}(k) \neq 2$. Show that $\mathfrak{go}_n(Q_n, k) \subseteq \mathfrak{sl}_n(k)$.

Exercise 4. (Schriftlich) Let H be an abelian Lie algebra and V be an H-module (both over $k = \mathbb{C}$). Let $H^* = \mathrm{Hom}(H, \mathbb{C})$ be the dual vector space. For $\lambda \in H^*$ define

$$V_\lambda := \{v \in V \mid h.v = \lambda(h)v \text{ for all } h \in H\}.$$

Show that V_λ is a subspace of V. Now let $r \geq 1$ and $\lambda, \lambda_1, \ldots, \lambda_r \in H^*$. Assume that $0 \neq v \in V_\lambda$ and $v \in \sum_{1 \leq i \leq r} V_{\lambda_i}$. Then show that $\lambda = \lambda_i$ for some i.

[*Hint.* First use Exercise 4(b) on Sheet 3 to show that, if $\mu_1, \ldots, \mu_m \in H^*$ are pairwise distinct, then there exists some $h \in H$ such that $\mu_i(h) \neq \mu_j(h)$ for all $i \neq j$; then remember the argument from Linear Algebra for proving that eigenvectors of a linear map corresponding to distinct eigenvalues are linearly independent.]

Abgabe: bis Dienstag, 12.5., 17:00 Uhr.