# 2. Exercise Sheet "Lie algebras and Chevalley groups"

Professor Meinolf Geck, SoSe 2020

**Exercise 1.** Let $L$ be a Lie algebra over a field $k$. Assume that $\dim L = 2$ and that $L$ is not abelian. Show that $L$ has a basis $\{x, y\}$ such that $[x, y] = y$; in particular, $\langle y \rangle_k$ is an ideal of $L$ and so $L$ is not simple. Show that $L$ is isomorphic to the following Lie subalgebra of $\mathfrak{gl}_2(k)$:

$$\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \;\middle|\; a, b \in k \right\}.$$

**Exercise 2.** (Schriftlich) Let $k$ be a field and $L = \mathfrak{sl}_2(k)$ (the Lie algebra of $2 \times 2$-matrices with trace 0, with Lie product $[A, B] = AB - BA$). Then $\dim L = 3$ and $L$ has a basis $\{h, e, f\}$ where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Check that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.
Show that $L$ is simple if $\mathrm{char}(k) \neq 2$. What happens if $\mathrm{char}(k) = 2$?

**Exercise 3.** (Schriftlich) We define

$$L = \left\{ \begin{pmatrix} 0 & t & x \\ -t & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; t, x, y \in k \right\} \quad \text{and} \quad L_\delta := \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & c & a\delta \end{pmatrix} \;\middle|\; a, b, c \in k \right\},$$

where $0 \neq \delta \in k$ is fixed. Show that $L$ and $L_\delta$ are subalgebras of $\mathfrak{gl}_3(k)$. Are $L$ and $L_\delta$ simple? Show that, if $L_\delta \cong L_{\delta'}$, then $\delta = \delta'$ or $\delta^{-1} = \delta'$. Hence, if $|k| = \infty$, then there are infinitely many pairwise non-isomorphic Lie algebras of dimension 3. (See Chap. 3 of the book by Erdmann–Wilson for further examples.)

[*Hint.* A useful tool to check that two Lie algebras cannot be isomorphic is as follows. Let $L_1, L_2$ be finite-dimensional Lie algebras over $k$. Let $\varphi\colon L_1 \to L_2$ be an isomorphism. Show that $\varphi \circ \mathrm{ad}_{L_1}(x) = \mathrm{ad}_{L_2}(\varphi(x)) \circ \varphi$ for $x \in L_1$. Deduce that $\mathrm{ad}_{L_1}(x)\colon L_1 \to L_1$ and $\mathrm{ad}_{L_2}(\varphi(x))\colon L_2 \to L_2$ must have the same characteristic polynomial. Try to apply this with the element $x \in L_\delta$ where $a = 1$, $b = c = 0$.]

**Exercise 4.** Let $L = \mathbb{R}^3$ with Lie bracket given by the vector product $[x, y] = x \times y$.
Show that $L$ is simple. By Exercise 2, we also know that $\mathfrak{sl}_2(\mathbb{R})$ is simple. Is $L \cong \mathfrak{sl}_2(\mathbb{R})$?

**Abgabe:** bis Dienstag, 28.4., 17:00 Uhr.