12. Exercise Sheet "Lie algebras and Chevalley groups"

Professor Meinolf Geck, SoSe 2020

Exercise 1. Define matrices $e_1, e_2, f_1, f_2 \in \mathfrak{gl}_7(\mathbb{C})$ as follows:

$$e_1 : \begin{pmatrix} . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}, \qquad e_2 : \begin{pmatrix} . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & 2 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & . \end{pmatrix},$$

$$f_1 : \begin{pmatrix} . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . \end{pmatrix}, \qquad f_2 : \begin{pmatrix} . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 2 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & 1 & . \end{pmatrix}.$$

(A dot . stands for 0.) Use a computer to verify that

$$h_1 := [e_1, f_1] = \operatorname{diag}(0, 1, -1, 0, 1, -1, 0),$$
$$h_2 := [e_2, f_2] = \operatorname{diag}(1, -1, 2, 0, -2, 1, -1),$$

and that the Chevalley relations (Ch1), (Ch2) hold with respect to the generalised Cartan matrix A of type G_2 (as in Table 4). Deduce that $L = \langle e_1, e_2, f_1, f_2 \rangle_{\text{alg}} \subseteq \mathfrak{gl}_7(\mathbb{C})$ is a simple Lie algebra of type G_2.

Exercise 2. (Schriftlich)
Let L be a simple Lie algebra of Cartan–Killing type. Let $\mathbf{B} = \{h_j^+ \mid j \in I\} \cup \{\mathbf{e}_\alpha^+ \mid \alpha \in \Phi\}$ be Lusztig's basis. Let $i \in I$ and $m \geq 1$ be an integer. Show that the matrices (with respect to $\mathbf{B}$) of the linear maps $\frac{1}{m!}\operatorname{ad}_L(e_i)^m \colon L \to L$ and $\frac{1}{m!}\operatorname{ad}_L(f_i)^m \colon L \to L$ have all their entries in $\mathbb{Z}$. (See again the proof of Theorem 3.5.1.) Check that $\operatorname{ad}_L(e_i)^4 = \operatorname{ad}_L(f_i)^4 = 0$.

Exercise 3. (Schriftlich) Let k be a field and $R = k[X_1, \ldots, X_n]$ the polynomial ring in n indeterminates.

(a) Assume that $|k| = \infty$. Let $f \in R$. Show that $f = 0$ if and only if $f(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in k$.

(b) Assume that k is algebraically closed. Let $f \in R$ be non-constant, where $n \geq 2$. Show that there are infinitely many tuples $(x_1, \ldots, x_n) \in k^n$ such that $f(x_1, \ldots, x_n) = 0$.

Abgabe: bis Donnerstag, 16.7., 11:00 Uhr.