11. Exercise Sheet "Lie algebras and Chevalley groups"

Professor Meinolf Geck, SoSe 2020

Exercise 1. Let A be an indecomposable generalized Cartan matrix of type (FIN). Let $\bar{A} \in M_I(\mathbb{Z})$ be the matrix with (i,j)-entry $|a_{ij}|$ for $i,j \in I$. Show that $\det(A) = \det(\bar{A})$.

[One possibility is to prove this case by case, for each of the matrices of type $A_n, B_n, \ldots, E_8$. Can you find a general argument?]

Exercise 2. Let A be an indecomposable generalized Cartan matrix of type (FIN). Let $\alpha_0 \in \Phi$ be the highest root (see Remark 3.2.10). Check that α_0 is always a long root.

[Again, this can be proved case by case. Can you find a general argument?]

Exercise 3. (Schriftlich) Let $R = \mathbb{C}[T, T^{-1}]$ be the ring of Laurent polynomials over $\mathbb{C}$ with indeterminate T. We consider the Lie algebra

$$L = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a,b,c \in R \right\} \quad \bigl(= \mathfrak{sl}_2(R)\bigr),$$

with the usual Lie bracket for matrices. A vector space basis of L is given by $\{T^k e_1, T^l h_1, T^m f_1 \mid k,l,m \in \mathbb{Z}\}$, where we set as usual:

$$e_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with relations $[e_1, f_1] = h_1$, $[h_1, e_1] = 2e_1$, $[h_1, f_1] = -2f_1$. Now set

$$e_2 := Tf_1, \qquad h_2 := -h_1, \qquad f_2 := T^{-1}e_1.$$

(a) Show that (Ch0) and the Chevalley relations (Ch1), (Ch2) hold with respect to the matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad \text{(affine type } \tilde{A}_1 \text{ in Table 5).}$$

(b) Let $H = \langle h_1, h_2 \rangle_{\mathbb{C}} \subseteq L$, $N^+ = \langle e_1, e_2 \rangle_{\text{alg}} \subseteq L$ and $N^- = \langle f_1, f_2 \rangle_{\text{alg}} \subseteq L$. Explicitly determine the subalgebras $N^+ \subseteq L$ and $N^- \subseteq L$. Show that $L = N^+ \oplus H \oplus N^-$.

Abgabe: bis Donnerstag, 9.7., 11:00 Uhr.