10. Exercise Sheet “Lie algebras and Chevalley groups”

Professor Meinolf Geck, SoSe 2020

Exercise 1. (Schriftlich) Let A be an indecomposable generalized Cartan matrix of type (FIN). Then $\det(A) \neq 0$ and we can form A^{-1}. Use condition (FIN) to show that all entries of A^{-1} are strictly positive rational numbers. Work out some examples explicitly.

Exercise 2. Not all the details of the proof of Theorem 3.1.12 are given in the lecture notes. Work out the details for the arguments mentioned in the last six lines of the proof of Theorem 3.1.12.

Exercise 3. The type of an indecomposable matrix A satisfying (C1) and (C2) in Section 3.1 can also be characterised in terms of the eigenvalues of A, as follows.

Choose any $c \in \mathbb{R}_{\geq 0}$ such that all diagonal entries of $B := c\,\mathrm{id}_I - A$ are ≥ 0. Then, by a weak form of the *Frobenius–Perron Theorem* (see, e.g., §8.2 in D. Serre, *Matrices: Theory and applications, 2nd edition*, Graduate Texts in Mathematics, 216, Springer-Verlag, New York, 2002), B has at least one real eigenvalue; furthermore, if μ_0 is the largest real eigenvalue, then there exists a corresponding eigenvector $v \in \mathbb{R}^I$ such that $v \geq 0$.

Deduce that $A = c\,\mathrm{id}_I - B$ has real eigenvalues. Show that $\lambda_0 = c - \mu_0$ is the smallest real eigenvalue of A and we still have $Av = (c - \mu_0)v = \lambda_0 v$. Then show:

$$A \text{ of type (FIN)} \;\Leftrightarrow\; \lambda_0 > 0, \qquad A \text{ of type (AFF)} \;\Leftrightarrow\; \lambda_0 = 0, \qquad A \text{ of type (IND)} \;\Leftrightarrow\; \lambda_0 < 0.$$

Work out some examples explicitly.

Abgabe: bis Donnerstag, 2.7., 11:00 Uhr.