1. Exercise Sheet "Lie algebras and Chevalley groups"

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Let k be a field. A k-algebra is just a vector space A together with a bilinear map $A \times A \to A$, $(a,b) \mapsto ab$, which defines a "product" on A. No further assumptions are implied. If the product is associative (or commutative or ...), then we will say this explicitly.

Exercise 1. Let A be a k-algebra. A linear map $d\colon A \to A$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. Let $\mathrm{Der}(A)$ be the set of all derivations of A.

(a) Show that $\mathrm{Der}(A)$ is a subspace of $\mathrm{End}(V)$; furthermore, if $d, d' \in \mathrm{Der}(A)$, then $[d, d'] := d \circ d' - d' \circ d \in \mathrm{Der}(A)$.

(b) Let $d\colon A \to A$ be a derivation. Show that, for any $n \geq 1$, we have the *Leibniz rule*

$$d^n(xy) = \sum_{i=0}^{n} \binom{n}{i} d^i(x) d^{n-i}(y) \qquad \text{for all } x, y \in A.$$

(c) Let $A = k[T, T^{-1}]$, the Laurent polynomial ring where T is an indeterminate. Determine a vector space basis of $\mathrm{Der}(A)$. (For example, the formal derivative $d\colon A \to A$, $T^n \mapsto nT^{n-1}$, is a derivation.)

Exercise 2. Let A be a k-algebra and $X \subseteq A$ be a subset. Then set $\hat{X} := \bigcup_{n \geq 0} X_n$ where the subsets $X_n \subseteq A$ are inductively defined by $X_1 := X$ and

$$X_n := \{x \cdot y \mid x \in X_i, y \in X_{n-i} \text{ for } 1 \leq i \leq n-1\} \quad \text{for } n \geq 2.$$

Thus, the elements in X_n are obtained by taking the iterated product, in any order, of n elements of X. We call the elements of X_n *monomials* in X (of level n). For example, if $X = \{x, y, z\}$, then $((z \cdot (x \cdot y)) \cdot z) \cdot ((z \cdot y) \cdot (x \cdot x))$ is a monomial of level 8 and, in general, we have to respect the parentheses in working with such products.

(a) We denote by $\langle X \rangle_{\mathrm{alg}} \subseteq A$ the subspace of A which is spanned by $\hat{X}$. Show that $\langle X \rangle_{\mathrm{alg}}$ is a subalgebra of A and that this is the smallest subalgebra which contains X.

(b) Let A be a Lie algebra. Let $V \subseteq A$ be a subspace such that $x \cdot v \in V$ for $x \in X$ and $v \in V$. Then show that $a \cdot v \in V$ and $v \cdot a \in V$ for $a \in \langle X \rangle_{\mathrm{alg}}$ and $v \in V$. Furthermore, if $X \subseteq V$, then $\langle X \rangle_{\mathrm{alg}} \subseteq V$.

(c) Let A be a Lie algebra and $I := \langle X \rangle_{\mathrm{alg}} \subseteq A$. Assume that $a \cdot x \in I$ for all $x \in X$, $a \in A$. Then show that I is an ideal of A.

Exercise 3. This is a reminder of a basic result from Linear Algebra. Let V be a vector space and $\varphi\colon V \to V$ be a linear map. Let $v \in V$. We say that φ is *locally nilpotent* at v if there exists some $d \geq 1$ (which may depend on v) such that $\varphi^d(v) = 0$. We say that φ is *nilpotent* if $\varphi^d = 0$ for some $d \geq 1$. Assume now that $\dim V < \infty$.

(a) Let $X \subseteq V$ be a subset such that $V = \langle X \rangle_k$. Assume that φ is locally nilpotent at every $v \in X$. Show that φ is nilpotent.

(b) Assume that φ is nilpotent. Show, as directly and simply as possible, that there is a basis B of V such that the matrix of φ with respect to B is triangular with 0 on the diagonal; in particular, we have $\varphi^{\dim V} = 0$ and the trace of φ is 0.