Zusatzaufgabe, Blatt 14

i) $|C| = 2 = \dim \mathbb{R}^2$. Nach Satz 4.9 gilt:

C ist eine Basis von $\mathbb{R}^2$, falls C l.u.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow{Z_2 \to Z_2 - Z_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow r = 2 = |C| \Rightarrow C \text{ l.u.}$$

Alternativ: Seien $\lambda_1, \lambda_2 \in \mathbb{R}$ mit $\lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\left.\begin{array}{l} \lambda_1 = 0 \\ \lambda_1 + \lambda_2 = 0 \end{array}\right\} \Rightarrow \lambda_1 = \lambda_2 = 0 \Rightarrow C \text{ l.u.}$$

$|C'| = 3 = \dim \mathbb{R}^3$. Nach Satz 4.9 gilt:

C' ist eine Basis von $\mathbb{R}^3$, falls C' l.u.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{Z_2 \to Z_2 - Z_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{Z_3 \to Z_3 - Z_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow$$

$r = 3 = |C'| \Rightarrow C'$ l.u.

Alternativ: Seien $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ mit $\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{cases} \lambda_1 + \lambda_3 = 0 \\ \lambda_1 + \lambda_2 = 0 \\ \lambda_2 + \lambda_3 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -\lambda_3 \\ \lambda_2 = -\lambda_1 = \lambda_3 \\ \lambda_3 + \lambda_3 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -\lambda_3 \\ \lambda_2 = \lambda_3 \\ \lambda_3 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases}$$

$\Rightarrow C'$ l.u.

b) $\begin{pmatrix}1\\1\end{pmatrix} = 1\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} = 1\,e_1 + 1\,e_2$

$\begin{pmatrix}0\\1\end{pmatrix} = 0\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} = 0e_1 + 1\,e_2$

Deshalb: $M_B^C(id_{\mathbb{R}^2}) = \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$

$\begin{pmatrix}1\\1\\0\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\1\end{pmatrix} = 1\,e_1 + 1\,e_2 + 0\,e_3$

$\begin{pmatrix}0\\1\\1\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix} = 0\,e_1 + 1\,e_2 + 1\,e_3$

$\begin{pmatrix}1\\0\\1\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix} = 1\,e_1 + 0\,e_2 + 1\,e_3$

$M_{B'}^{C'}(id_{\mathbb{R}^3}) = \begin{pmatrix}1 & 0 & 1\\1 & 1 & 0\\0 & 1 & 1\end{pmatrix}$

$M_B^C(id_{\mathbb{R}^2})^{-1} \underset{\text{Beispiel 2.11}}{=} M_C^B(id_{\mathbb{R}^2})$

$e_1 = \begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \left.\begin{matrix}a=1\\a+b=0\end{matrix}\right\} \begin{matrix}a=1\\b=-1\end{matrix}$

$e_2 = \begin{pmatrix}0\\1\end{pmatrix} = c\begin{pmatrix}1\\1\end{pmatrix} + d\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \left.\begin{matrix}c=0\\c+d=1\end{matrix}\right\} \begin{matrix}c=0\\d=1\end{matrix}$

$M_C^B(id_{\mathbb{R}^2}) = \begin{pmatrix}a & c\\b & d\end{pmatrix} = \begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}$

Ähnlich: $M_{B'}^{C'}(id_{\mathbb{R}^3})^{-1} = M_{C'}^{B'}(id_{\mathbb{R}^3})$

$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \begin{cases} a+c=1 \\ a+b=0 \\ b+c=0 \end{cases} \Rightarrow \begin{cases} a = 1/2 \\ b = -1/2 \\ c = 1/2 \end{cases}$$

$$e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} = d\begin{pmatrix}1\\1\\0\end{pmatrix} + e\begin{pmatrix}0\\1\\1\end{pmatrix} + f\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \begin{cases} d+f=0 \\ d+e=1 \\ e+f=0 \end{cases} \Rightarrow \begin{cases} d = 1/2 \\ e = 1/2 \\ f = -1/2 \end{cases}$$

$$e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} = g\begin{pmatrix}1\\1\\0\end{pmatrix} + h\begin{pmatrix}0\\1\\1\end{pmatrix} + i\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \begin{cases} g+i=0 \\ g+h=0 \\ h+i=1 \end{cases} \Rightarrow \begin{cases} g = -1/2 \\ h = 1/2 \\ i = 1/2 \end{cases}$$

$$M_{C'}^{B'}(id_{\mathbb{R}^3}) = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

c) $\phi_1\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} = 2e_1 + 1e_2$

$\phi_1\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix} = 3e_1 + (-2)e_2$

Deshalb: $M_B^B(\phi_1) = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$

$$M_C^B(\phi_1) = M_C^B(id_{\mathbb{R}^2}) \cdot M_B^B(\phi_1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \end{pmatrix}$$

<u>Alternativ</u>: $\phi_1\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} + (-1)\begin{pmatrix}0\\1\end{pmatrix}$

$\phi_1\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix} + (-5)\begin{pmatrix}0\\1\end{pmatrix}$

Deshalb: $M_C^B(\phi_1) = \begin{pmatrix} 2 & 3 \\ -1 & -5 \end{pmatrix}$

$$\phi_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix}+0\begin{pmatrix}0\\1\\0\end{pmatrix}+2\begin{pmatrix}0\\0\\1\end{pmatrix} = 1e_1+0e_2+2e_3$$

$$\phi_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\end{pmatrix}-1\begin{pmatrix}0\\1\\0\end{pmatrix}+1\begin{pmatrix}0\\0\\1\end{pmatrix} = 0e_1+(-1)e_2+1e_3$$

Deshalb: $M^{C}_{B'}(\phi_2) = \begin{pmatrix}1&0\\0&-1\\2&1\end{pmatrix}$

$$\phi_3\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix}+1\begin{pmatrix}0\\1\end{pmatrix} = 2e_1+1e_2$$

$$\phi_3\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} = 1\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix} = 1e_1+2e_2$$

$$\phi_3\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = 1\begin{pmatrix}1\\0\end{pmatrix}+1\begin{pmatrix}0\\1\end{pmatrix} = 1e_1+1e_2$$

Deshalb: $M^{C'}_{B} = \begin{pmatrix}2&1&1\\1&2&1\end{pmatrix}$

Es ist $\phi_4\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix} = 0e_1+(-1)e_2+1e_3$

$\phi_4\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} = 0e_1+1e_2+1e_3$

$\phi_4\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} = 1e_1+0e_2+1e_3$

$\Rightarrow M^{B'}_{B'}(\phi_4) = \begin{pmatrix}0&0&1\\-1&1&0\\1&1&1\end{pmatrix}$

Es ist $M^{B'}_{C'}(\phi_4) = M^{B'}_{C'}(id_{\mathbb{R}^3})\cdot M^{B'}_{B'}(\phi_4)$

$$\overset{(b)}{=} \frac{1}{2}\begin{pmatrix}1&1&-1\\-1&1&1\\1&-1&1\end{pmatrix}\cdot\begin{pmatrix}0&0&1\\-1&1&0\\1&1&1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}-2&1&0\\0&1&0\\2&-1&2\end{pmatrix}$$

Alternativ

$$\Phi_4\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \begin{cases} a+c=1 \\ a+b=-1 \\ b+c=1 \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=0 \\ c=1 \end{cases}$$

$$\Phi_4\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} = d\begin{pmatrix}1\\1\\0\end{pmatrix} + e\begin{pmatrix}0\\1\\1\end{pmatrix} + f\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \begin{cases} d+f=0 \\ d+e=1 \\ e+f=0 \end{cases} \Rightarrow \begin{cases} d=1/2 \\ e=1/2 \\ f=-1/2 \end{cases}$$

$$\Phi_4\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} = g\begin{pmatrix}1\\1\\0\end{pmatrix} + h\begin{pmatrix}0\\1\\1\end{pmatrix} + i\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \begin{cases} g+i=1 \\ g+h=0 \\ h+i=1 \end{cases} \Rightarrow \begin{cases} g=0 \\ h=0 \\ i=1 \end{cases}$$

$$M^{B'}_{C'}(\Phi_4) = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} -1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 1 & -1/2 & 1 \end{pmatrix} = 1/2\begin{pmatrix} -2 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix}$$

(d) Seien $A := M^B_B(\Phi_1) \overset{(c)}{=} \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$

$B := M^{B'}_{B'}(\Phi_4) \overset{(c)}{=} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$\mu_A(X) = X^n + a_{n-1}X^{n-1} + \dots + a_1 X + a_0$$

Fall 1: $n=1 \Rightarrow \mu_A(x) = X + a_0$

$\mu_A(A) = 0_{2\times 2} \Rightarrow A + a_0 I_2 = \begin{pmatrix} 2+a_0 & 3 \\ * & * \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ↯

Fall 2: $n=2 \Rightarrow \mu_A(x) = X^2 + a_1 X + a_0$

$$\mu_A(A) = 0_{2\times 2} \Rightarrow A^2 + a_1 A + a_0 I_2 = \begin{pmatrix} 7 + 2a_1 + a_0 & 3a_1 \\ a_1 & 7 - 2a_1 + a_0 \end{pmatrix} = 0_{2\times 2}$$

$$\Rightarrow \begin{cases} a_1 = 0 \\ 7 + 2a_1 + a_0 = 0 \\ 7 - 2a_1 + a_0 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = -7 \end{cases} \Rightarrow \mu_A(x) = X^2 - 7$$

Die Eigenwerte von A sind die Nullstellen von $\varphi_A(x)$.

$\Rightarrow$ A hat zwei Nullstellen, nämlich $-\sqrt{7}$ und $\sqrt{7}$.

$$\mu_B(x) = X^n + b_{n-1}X^{n-1} + \dots + b_0$$

<u>Fall 1</u>: $n=1 \Rightarrow \mu_B(x) = X + b_0$.

$$\mu_B(B) = B + b_0 I_3 = \begin{pmatrix} b_0 & 0 & 1 \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ↯$$

<u>Fall 2</u>: $n=2 \Rightarrow \mu_B(x) = X^2 + b_1 X + b_0$

$$\mu_B(B) = B^2 + b_1 B + b_0 I_3 \Rightarrow \begin{pmatrix} * & * & * \\ * & * & -1 \\ * & * & * \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ↯$$

$$B^2 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

<u>Fall 3</u>: $n=3 \Rightarrow \mu_B(x) = X^3 + b_2 X^2 + b_1 X + b_0$

$$\mu_B(B) = B^3 + b_2 X^2 + b_1 X + b_0 I_3 = \begin{pmatrix} 1 + b_2 + b_0 & 0 & 2 + b_2 + b_1 \\ -2 - b_2 - b_1 & 1 + b_2 + b_1 + b_0 & -2 - b_2 \\ 2 + b_2 + b_1 & 0 & 3 + 2b_2 + b_1 + b_0 \end{pmatrix}$$

$$B^3 = B^2 \cdot B = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -2 \\ 2 & 0 & 3 \end{pmatrix}$$

$$\mu_B(B) = 0_{3\times 3} \Rightarrow \begin{cases} b_0 + b_2 = -1 \\ b_0 + b_1 + b_2 = -1 \\ b_1 + b_2 = -2 \\ b_2 = -2 \\ b_0 + b_1 + 2b_2 = -3 \end{cases} \Rightarrow \begin{cases} b_1 = 0 \\ b_2 = -2 \\ b_0 = 1 \end{cases}$$

$$\Rightarrow \mu_B(X) = X^3 - 2X^2 + 1 = (X-1)(X^2 - X - 1)$$

Die Eigenwerte von B sind die Nullstellen von $\mu_B(X)$.

Deshalb hat B drei Eigenwerte, nämlich 1, $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$.