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Characters of finite Coxeter groups and Iwahori-Hecke algebras, by Meinolf Geck and Götz Pfeiffer, Oxford University Press, 2000, xv + 443 pp., \$110.00, ISBN 0-19-850250-8

The idea of an Iwahori-Hecke algebra originated in Iwahori's 1964 paper 'On the structure of a Hecke ring of a Chevalley group over a finite field' [22]. The finite Chevalley groups, such as $SL(n, q)$, are analogues over finite fields of the simple Lie groups. Such a Chevalley group $G(q)$ has a Borel subgroup $B(q)$ (the subgroup of triangular matrices in the case $SL(n, q)$), and one considers the $G(q)$-module V affording the representation induced from the unit representation of $B(q)$. The Hecke algebra H is the endomorphism algebra of V. Iwahori showed that the dimension of H is the order of the Weyl group W of $G(q)$ and that H has a basis T_w, $w \in W$, of elements satisfying the relations

$$T_w T_{w'} = T_{ww'} \quad \text{if } l(ww') = l(w) + l(w')$$

and also the quadratic relations

$$T_s^2 = (q-1)T_s + qT_1$$

for each simple reflection $s \in W$.

This result was generalized by Matsumoto [31] to the case in which $G(q)$ is a twisted group of Lie type. Instead of taking parameter q in all quadratic relations, one must take one parameter for each W-conjugate set of simple reflections.

In fact, one can consider Hecke algebras defined by analogous generators and relations for any Coxeter group W. Moreover, it is natural to consider also generic Hecke algebras in which the parameters which appear in the quadratic relations are replaced by indeterminates.

Algebras of the above types are nowadays called Iwahori-Hecke algebras. This is appropriate on historical grounds and is also useful in distinguishing these algebras from the Hecke algebras which appear in number theory. However, we shall continue to use the abbreviation 'Hecke algebras' for 'Iwahori-Hecke algebras' when this is convenient.

The structure of the Iwahori-Hecke algebra was clarified by Tits, who showed [4] that this algebra over $\mathbb{C}$ is isomorphic to the group algebra $\mathbb{C}W$. Tits used a deformation argument to prove this result, but no explicit isomorphism was known at this stage.

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Attention then turned to the representation theory of the Iwahori-Hecke algebra. Curtis, Iwahori and Kilmoyer in 1971 studied the representation of H analogous to the reflection representation of W [11]. Then Hoefsmit, in his Ph.D. thesis of 1974, determined a complete set of irreducible matrix representations of the Hecke algebras of classical type [20]. Appreciation of the relevance of Hecke algebras for the complex representation theory of finite reductive groups increased when it was shown by Howlett and Lehrer in 1980 [21] that the endomorphism algebra of the module induced from any cuspidal irreducible representation of a Levi subgroup, lifted to a parabolic subgroup, is isomorphic to a Hecke algebra. Thus the irreducible components of these induced modules are in natural bijection with the irreducible modules for the Hecke algebra, and thus with those of the corresponding Coxeter group. This reduces the study of arbitrary irreducible representations to that of cuspidal representations [10].

Radical new ideas on Iwahori-Hecke algebras appeared in the paper of Kazhdan and Lusztig, 'Representations of Coxeter groups and Hecke algebras', in 1979. In a paper of astonishing originality the authors showed that representations could be constructed from combinatorial objects called W-graphs and that W-graphs could be obtained from the Hecke algebra itself by means of a basis C_w of H, now known as the Kazhdan-Lusztig basis. When the basis C_w is expressed in terms of the natural basis T_w, one obtains the so-called Kazhdan-Lusztig polynomials. These polynomials have since turned out to be crucial in obtaining character formulae for irreducible modules in various contexts in representation theory. The coefficients of the Kazhdan-Lusztig polynomials have geometrical significance in connection with the intersection cohomology of Schubert varieties [23]. By making use of the ideas in this paper, Lusztig subsequently obtained [27] a canonical isomorphism between the Hecke algebra and the group algebra of the Coxeter group over the field $\mathbb{Q}(q^{1/2})$.

An interesting contribution to the representation theory of Hecke algebras of type A was made by Starkey in his Ph.D. thesis of 1975 [32]. Starkey obtained a square character table for the Hecke algebra which specialised to the character table of the symmetric group when q was replaced by 1. The problem in obtaining such a square character table for the Hecke algebra is that the irreducible characters of H do not take constant value on the basis elements T_w for w in a conjugacy class of W. It was shown by Starkey, however, that the characters do take constant value on those T_w for which w is of minimal length in its conjugacy class. The values on such elements T_w were used in his character table of H.

It was the attempt to generalize Starkey's results to Iwahori-Hecke algebras of arbitrary finite Coxeter groups which led Geck and Pfeiffer to the work on which the book under review is based. A systematic description of the conjugacy classes in Weyl groups had been given by Carter in 1972 [9]. This description was given in terms of the reflections in W, but did not single out any particular set of simple reflections. However, in order to define a square character table for the Hecke algebra, a new understanding of the relation between the conjugacy classes of W and the length function was necessary. Geck and Pfeiffer showed [19] that each element $w \in W$ can be transformed by a sequence of conjugations by simple reflections, not increasing the length at any stage, into an element of minimal length in its conjugacy class. By using this property they were able to show that the irreducible characters of H are constant on basis elements T_w for elements w of minimal length in a given conjugacy class, and thus to define the square character table of H.

It was shown by Geck and Michel [18] that, in fact, every conjugacy class of W contains at least one element of a very special kind, called a good element, a property defined in terms of the braid monoid. Normal forms in the braid monoid due to Deligne and Brieskorn-Saito are described in Geck and Pfeiffer's book, and the definition of good elements is related to the Brieskorn-Saito normal form.

Applications of the braid monoid to the theory of knots and links are also described in the book. Consider the Hecke algebra of the symmetric group S_n with quadratic relations $T_s^2 = uT_1 + vT_s$. Every knot or link in $\mathbb{R}^3$ gives rise to an element g in the braid group, and this element has an image $\bar{g}$ in the Hecke algebra. The value $\tau_n(\bar{g})$ of the Markov trace τ_n at this image is an element of $\mathbb{Z}[u, u^{-1}, v, v^{-1}]$. The particular case $u = 1$, $v = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ gives the Alexander polynomial; the case $u = t^2, v = t(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ gives the Jones polynomial; and the case $u = t^2$, $v = tx$ gives the HOMFLY polynomial.

Before embarking on a description of the character tables of the Iwahori-Hecke algebras, Geck and Pfeiffer provide an exposition of the representation theory of finite Coxeter groups. A key idea in obtaining the irreducible representations is the use of truncated induction from reflection subgroups. In particular we have Macdonald representations which are obtained by truncated induction from the sign representation. All irreducible representations of Coxeter groups of types A and B can be obtained as Macdonald representations. All irreducible representations of Weyl groups are of parabolic type, i.e. appear with multiplicity 1 in the character induced from the unit character of some parabolic subgroup. The irreducible characters of W fall into families in a way determined by the Kazhdan-Lusztig theory. This theory gives a decomposition of W into equivalence classes called two-sided cells, and there is one family of irreducible characters for each two-sided cell. Each family contains a canonical representative called the special representation in the family.

The Iwahori-Hecke algebras are examples of symmetric algebras, and Geck and Pfeiffer include a chapter on the representation theory of symmetric algebras before considering that of the Hecke algebras themselves. If H is a split symmetric algebra, the characters of its irreducible modules satisfy orthogonality relations analogous to those which hold for group algebras. These relations involve the Schur element c_V of an irreducible module V, which in the special case of a group algebra is equal to the order of the group divided by $\dim V$. One also obtains formulae for primitive idempotents in terms of the c_V.

In the case when H is an Iwahori-Hecke algebra, we have, for each irreducible character χ, the generic degree $D_\chi = P_W / c_\chi$ where P_W is the Poincaré polynomial of W and c_χ is the Schur element of χ. In the situation in which H is the endomorphism algebra of the module induced from the unit representation of a Borel subgroup $B(q)$ of $G(q)$, D_χ is the degree of the irreducible character of $G(q)$ corresponding to the irreducible character χ of H. The square character table $X(H)$ has entries $\chi(T_w)$ where w has minimal length in its conjugacy class, and it is shown how to obtain the values $\chi(T_w)$ for arbitrary w in terms of these. Starkey's rule is described for obtaining $X(H)$ when H is of type A_n. Hoefsmit's matrices representing generators of H are described when H is of type B_n. A beautiful formula is given for obtaining the character values of $H(B_n)$ recursively, which generalizes the Murnaghan-Nakayama formula for obtaining the character values of the symmetric group. Analogous results are described for Hecke algebras of type D_n, although the results are somewhat more complicated. For each Iwahori-Hecke algebra of exceptional type Geck and Pfeiffer have determined the character table by computational methods, implemented in the CHEVIE package of the computer algebra system GAP. The Kazhdan-Lusztig theory is described, and W-graphs giving each of the irreducible representations of H are given in the exceptional types H_3, H_4, F_4. The W-graphs in type F_4 are obtained here for the first time.

The generic degree D_χ of each irreducible character χ for each algebra H of exceptional type is obtained. This is done by finding by various methods linear equations having the D_χ as solutions. Although there is no a priori reason why this should be so, it turns out that, in the exceptional types H_3, H_4, F_4, E_6, E_7, E_8, these equations have a unique solution. When all the parameters in H are equal, the generic degrees D_χ are polynomials, but this is not always true in the case of unequal parameters.

The idea for obtaining the character values $\chi(T_w)$ is similar. Suppose again that H is of exceptional type and all parameters are powers of v^2 (this is no real restriction). Then we have $\chi(T_w) = \sum_j a_{\chi,j} v^j$ for certain real numbers $a_{\chi,j}$. One can find, using a variety of techniques, linear equations having the $a_{\chi,j}$ as solutions. Again this system of equations turns out to have a unique solution, thus enabling the $a_{\chi,j}$ to be determined. The character tables $X(H)$ of exceptional type are not printed in the book (the case E_8 in particular is of a formidable size) but are available electronically. (See also [14], [15].)

The character tables may be applied to a study of the modular characters of Iwahori-Hecke algebras. Suppose H is such an algebra with all parameters equal to v^2, and consider the specialisation in which v is replaced by a primitive $2e$-th root of unity ζ. One obtains a specialised algebra H_ζ, and there are only finitely many e for which H_ζ is not semisimple. For these values of e one obtains a decomposition matrix expressing the irreducible representations of H, when specialised, in terms of those of H_ζ. These decomposition matrices are obtained for each exceptional type and each e.

The book contains a useful appendix containing detailed information about conjugacy classes and irreducible characters of Coxeter groups of exceptional type, and generic degrees and blocks for the corresponding Iwahori-Hecke algebras.

One remarkable feature of the theory of Coxeter groups and Iwahori-Hecke algebras is the number of key properties which at present have no uniform proof and can only be proved in a case-by-case manner. We mention three examples of such properties. In the first place every element of a Weyl group is a product of two involutions. This is a key property in Carter's description of the conjugacy classes in the Weyl group [9]. Secondly, every element of a finite Coxeter group can be transformed into an element of minimal length in its conjugacy class by a sequence of conjugations by simple reflections such that the length does not increase at any stage. This property is basic to the Geck-Pfeiffer approach to the conjugacy classes of Coxeter groups. Thirdly, there are basic properties of Lusztig's a-function which at present have only case-by-case proofs. The a-function is an important invariant of irreducible characters of Coxeter groups. One cannot be satisfied with the theory of Coxeter groups until such case-by-case proofs are replaced by uniform proofs of a conceptual nature.

This is a well-written monograph with a high degree of originality. Both the concept of a square character table of a general Iwahori-Hecke algebra and the explicit determination of the character values in all cases are due to Geck and Pfeiffer. It is clear that Iwahori-Hecke algebras will continue at the forefront of interest in representation theory for the foreseeable future. For example the affine Hecke algebras (Hecke algebras of affine Weyl groups and extended affine Weyl groups) are of key importance for the representation theory of Chevalley groups over local fields, for the modular representation theory in equal characteristic of Chevalley groups over finite fields, and for the representation theory of quantized enveloping algebras, as well as in other contexts [1]. Moreover there are analogues of Hecke algebras called cyclotomic algebras, associated with complex reflection groups, for which a general theory is at present taking shape. Such cyclotomic algebras are of importance in understanding the modular representation theory of finite reductive groups in which the characteristic of the field is not equal to that of the group [6], [7], [8].

A further avenue for continuing work is the circle of ideas relating to Ariki's proof of the Lascoux-Leclerc-Thibon conjecture. This concerns the Iwahori-Hecke algebra of type A_n and its specialisation when its parameter is replaced by an eth root of unity. The decomposition matrix D_e can be obtained in terms of the canonical basis of a certain highest weight module for an affine Kac-Moody algebra. This result gives rise to an algorithm which will determine the decomposition matrix D_e. These results are described in Geck and Pfeiffer's book without proof. A survey article by Geck on this subject can be found in [16]. In fact Ariki's proof uses deep results on representations of affine Hecke algebras, canonical bases, and intersection cohomology [2]. Further investigation of this circle of ideas is sure to prove fruitful.

This volume by Geck and Pfeiffer on the representation theory of the Iwahori-Hecke algebras is therefore very timely. It will be welcomed by the mathematical community.

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