# James' Conjecture for Hecke Algebras of Exceptional Type

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"Everybody" knows that the irreducible representations of the symmetric group $\mathfrak{S}_n$ are parametrized by the partitions of $n$.

(Note: this is true for representations over a field of characteristic 0.)

Explicit model for irreducible representation labelled by $\lambda \vdash n$:

"Specht module" $S^\lambda$ (G. D. James 1970's).

- $S^\lambda$ defined over $\mathbb{Z}$;

  (standard basis $\{e_{\mathfrak{t}}\}$ where $\mathfrak{t}$ runs over the set of standard $\lambda$-tableaux).

- $\exists\ \beta^\lambda \colon S^\lambda \times S^\lambda \to \mathbb{Z}$ symmetric bilinear form such that

$$\beta^\lambda(\pi.e, \pi.e') = \beta^\lambda(e, e') \qquad \forall\ \pi \in \mathfrak{S}_n \quad \forall\ e, e' \in S^\lambda.$$

- $k$ any field $\quad\rightsquigarrow\quad S_k^\lambda = k \otimes_{\mathbb{Z}} S^\lambda$ module for $\mathfrak{S}_n$ over $k$, and

$$\beta_k^\lambda \colon S_k^\lambda \times S_k^\lambda \to k \quad \text{and} \quad \mathrm{rad}(\beta_k^\lambda) \subseteq S_k^\lambda \quad \text{(submodule)}.$$

Set $D_k^\lambda = S_k^\lambda/\mathrm{rad}(\beta_k^\lambda)$. Then

$$\mathrm{Irr}_k(\mathfrak{S}_n) = \{D_k^\lambda \mid \lambda \vdash n \text{ such that } \beta_k^\lambda \neq 0\}.$$

If $\mathrm{char}(k) = 0$, then $k[\mathfrak{S}_n]$ is semisimple and $D_k^\lambda = S_k^\lambda$ for all $\lambda \vdash n$.

If $\mathrm{char}(k) = p > 0$, then $\beta_k^\lambda \neq 0$ if and only if $\lambda$ is $p$-regular.

(Write $\lambda = (1^{n_1}, 2^{n_2}, 3^{n_3}, \ldots)$. Then $\lambda$ is $p$-regular if and only if $n_i < p$ for all $i$).

**Major open problem:**

Find explicit formulas for $\dim D_k^\lambda$ or, equivalently, for $\dim \mathrm{rad}(\beta_k^\lambda)$.

More precisely: Determine decomposition matrix

$$D = \Big([S_k^\lambda : D_k^\mu]\Big)_{\lambda \vdash n,\ \mu \vdash n\ p\text{-regular}}$$

R. Dipper + G. D. James (1980/90's): Generalisation to Hecke algebra of $\mathfrak{S}_n$ and application to modular representations of $\mathrm{GL}_n(\mathbb{F}_q)$.

$k$ any field, $q \in k^\times \quad \rightsquigarrow \quad H_n(k,q)$ Hecke algebra of $\mathfrak{S}_n$.

- associative algebra over $k$, basis $\{T_w \mid w \in \mathfrak{S}_n\}$;
- $T_{s_i}^2 = qT_1 + (q-1)T_{s_i}$ for $s_i = (i, i+1)$ where $1 \leqslant i \leqslant n-1$;
  $T_w = T_{s_{i_1}} \cdots T_{s_{i_l}}$ if $w = s_{i_1} \cdots s_{i_l}$ minimal expression.

For $\lambda \vdash n$, there is a Specht module $S_q^\lambda$ of $H_n(k,q)$, equipped with an $H_n(k,q)$-invariant bilinear form $\beta_q^\lambda$. Set $D_q^\lambda = S_q^\lambda/\mathrm{rad}(\beta_q^\lambda)$. Then

$$\mathrm{Irr}(H_n(k,q)) = \{D_q^\lambda \mid \lambda \vdash n \text{ such that } \beta_q^\lambda \neq 0\}.$$

Set $e = \min\{i \geqslant 2 \mid 1 + q + q^2 + \cdots + q^{i-1} = 0\}$.

($e = \infty$ if no such $i$ exists.)

- If $e > n$, then $H_n(k,q)$ is semisimple and $D_q^\lambda = S_q^\lambda$ for all $\lambda \vdash n$.
- $\beta_q^\lambda \neq 0$ if and only if $\lambda$ is $e$-regular.

**Major open problem ("$q$-version"):**

Find explicit formulas for $\dim D_q^\lambda$ or, equivalently, for $\dim \operatorname{rad}(\beta_q^\lambda)$.

More precisely: Determine decomposition matrix

$$D_q = \Big([S_q^\lambda : D_q^\mu]\Big)_{\lambda\vdash n,\ \mu\vdash n\ e\text{-regular}}$$

Note: If $q = 1$ and $\operatorname{char}(k) = \ell > 0$, then $H_n(k, q) = k[\mathfrak{S}_n]$, $e = \ell$ and "1-version" is the "major open problem" for $\operatorname{Irr}_k(\mathfrak{S}_n)$.

**James' Conjecture**, Proc. London Math. Soc. **60** (1990)

Assume that $e < \infty$, $\operatorname{char}(k) = \ell > 0$ and $e\ell > n$. Then the decomposition matrix $D_q$ only depends on $e$. More precisely, consider the Hecke algebra $H_n(\mathbb{C}, \zeta_e)$, where $\zeta_e = \sqrt[e]{1} \in \mathbb{C}$. Then

$$[S_q^\lambda : D_q^\mu] = [S_{\zeta_e}^\lambda : D_{\zeta_e}^\mu] \qquad \forall\ \lambda \vdash n,\ \mu \vdash n\ e\text{-regular}.$$

## Remark 1.

A. Lascoux, B. Leclerc and J. Y. Thibon (1996):
Conjecture which would give a purely combinatorial algorithm for computing the numbers $[S^\lambda_{\zeta_e} : D^\mu_{\zeta_e}]$. Proved by S. Ariki (1997).

## Remark 2.

Let $q = 1$ and $\mathrm{char}(k) = \ell > 0$. Then $H_n(k, q) = k[\mathfrak{S}_n]$ and $e = \ell$.

$$\text{James' conjecture} \quad \Rightarrow \quad \text{If } \ell^2 > n, \text{ then } [S^\lambda_1 : D^\mu_1] = [S^\lambda_{\zeta_\ell} : D^\mu_{\zeta_\ell}],$$

and these numbers can be computed by the LLT-algorithm.

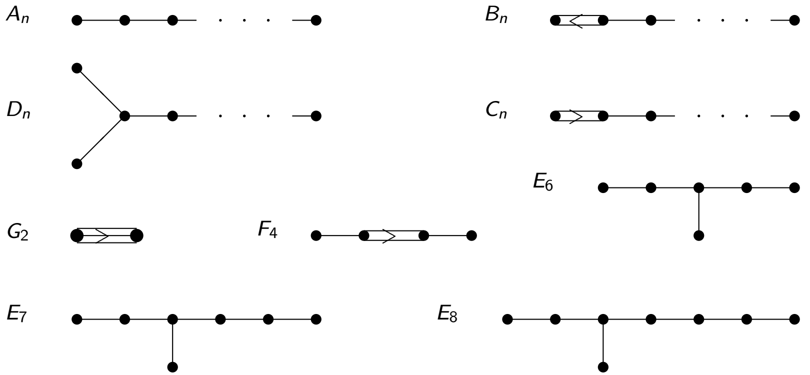
## Remark 3.

M. G. (1990) formulated a version of James' conjecture for Hecke algebras of *any* type and proved that it is true if $\ell \gg 0$. Thus, James' conjecture really is about finding the correct bound on $\ell$.

## Outline of the talks

- The formulation of the general version of James' conjecture is based on a general theory of Specht modules for Hecke algebras. (This approach is different from the one in my old article from 1990.)
- The new approach leads to a straightforward algorithm for verifying James' conjecture for a given Hecke algebra.
- We will use this to verify the conjecture for $W$ of exceptional type, including $E_8$.
- Nevertheless, carrying out the steps in the algorithm requires the use of a number of sophisticated computer algebra packages and a lot of memory and cpu time (more than 6 months).

$W$ = Weyl group associated to one of the following diagrams:




Generating set $S$ (in bijection with nodes); $s^2 = 1$ for all $s \in S$.

"Braid" relations: For $s \neq t$ in $S$, we have $(st)^m = 1$ where

$$m = 2, 3, 4 \text{ or } 6 \quad \text{if } s, t \text{ are joined by } 0, 1, 2 \text{ or } 3 \text{ edges.}$$

$\mathcal{H} = \mathcal{H}_A(W)$ generic Iwahori–Hecke algebra over $A = R[u^{1/2}, u^{-1/2}]$ (where $R \subseteq \mathbb{C}$ is any subring).

- free $A$-module, basis $\{T_w \mid w \in W\}$;
- associative multiplication

$$T_w = T_{s_1} \cdots T_{s_l} \quad \text{if } w = s_1 \cdots s_l \ (s_i \in S) \text{ minimal expression;}$$
$$T_s^2 = uT_1 + (u-1)T_s \qquad \text{for } s \in S.$$

Specialisation: ring homomorphism $A \to k$ ($k$ field), $u \mapsto \xi \in k$.

$\mathcal{H}_{k,\xi} = k \otimes_A \mathcal{H}$ associative algebra over $k$; basis $\{T_w \mid w \in W\}$.

$$T_s^2 = \xi T_1 + (\xi - 1) T_s \text{ for } s \in S.$$

$\xi = 1 \rightsquigarrow T_s^2 = T_1$ for all $s \in S$ and so $\mathcal{H}_{k,1} = k[W]$ (group algebra).

**Problem:** Determine $\mathrm{Irr}(\mathcal{H}_{k,\xi})$ for various $k, \xi$.

Main application:

$G = G(\mathbb{F}_q)$ finite group of Lie type with Weyl group $W$. Examples:

$G = \mathrm{GL}_n(\mathbb{F}_q)$ has Weyl group $W(A_{n-1}) \cong \mathfrak{S}_n$.

$G = \mathrm{SO}_{2n+1}(\mathbb{F}_q)$ has Weyl group $W(C_n) \cong W(B_n)$.

$B \subseteq G$ Borel subgroup (normalizer of a $p$-Sylow where $q = p^f$);

$k$ (sufficiently large) field of characteristic $\ell \geqslant 0$ where $\ell \nmid q$.

**Theorem.** N. Iwahori ($\sim$ 1964) + R. Dipper ($\sim$ 1990)

$\mathrm{Irr}(\mathcal{H}_{k,\bar{q}})$ is naturally in bijection with the "unipotent principal series $k$-representations" of $G$, *i.e.*, those irreducible representations of $G$ over $k$ which admit non-zero vectors fixed by $B$.

Let $H$ be any associative algebra over a commutative ring $A$.

A general theory of "Specht modules" is provided by the "cell data" introduced by J. Graham and G. Lehrer (1996).

### A "cell datum" $(\Lambda, M, C, *)$ for $H$ has to satisfy:

- $\Lambda$ is a partially ordered set, $\{M(\lambda) \mid \lambda \in \Lambda\}$ are finite sets and

$$\{C^{\lambda}_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \Lambda \text{ and } \mathfrak{s},\mathfrak{t} \in M(\lambda)\} \quad \text{is an } A\text{-basis of } H.$$

- $*\colon H \to H$ is an $A$-linear anti-involution such that

$$(C^{\lambda}_{\mathfrak{s},\mathfrak{t}})^* = C^{\lambda}_{\mathfrak{t},\mathfrak{s}} \text{ for all } \lambda \in \Lambda \text{ and } \mathfrak{s},\mathfrak{t} \in M(\lambda).$$

- For $h \in H$: $\quad hC^{\lambda}_{\mathfrak{s},\mathfrak{t}} = \sum_{\mathfrak{s}' \in M(\lambda)} r_h(\mathfrak{s}',\mathfrak{s})\, C^{\lambda}_{\mathfrak{s}',\mathfrak{t}} + \text{"lower terms"}$,
  - where "lower terms" means: combination of $C^{\mu}_{\mathfrak{s}'',\mathfrak{t}''}$ where $\mu < \lambda$,
  - and $r_h(\mathfrak{s}',\mathfrak{s}) \in A$ is independent of $\mathfrak{t}$.

Fix $\lambda \in \Lambda$. Let $W^\lambda$ be a free $A$-module with basis $\{C_\mathfrak{s} \mid \mathfrak{s} \in M(\lambda)\}$.

Left action of $H$: $h.C_\mathfrak{s} = \sum_{\mathfrak{s}' \in M(\lambda)} r_h(\mathfrak{s}', \mathfrak{s})\, C_{\mathfrak{s}'}$.

$H$-equivariant bilinear form $\phi^\lambda \colon W^\lambda \times W^\lambda \to A$, defined by:

$$C^\lambda_{\mathfrak{s}',\mathfrak{s}} C^\lambda_{\mathfrak{t},\mathfrak{t}'} = \phi^\lambda(C_\mathfrak{s}, C_\mathfrak{t})\, C^\lambda_{\mathfrak{s}',\mathfrak{t}'} \quad + \quad \text{"lower terms"}.$$

Specialisation $A \to k$ ($k$ field) $\rightsquigarrow$ $k$-algebra $H_k = k \otimes_A H$ and cell modules $W_k^\lambda = k \otimes_A W^\lambda$, with induced bilinear forms $\phi_k^\lambda$.

**Theorem.** Graham–Lehrer ($\sim$ 1996)

Set $L_k^\lambda = W_k^\lambda / \operatorname{rad}(\phi_k^\lambda)$ for $\lambda \in \Lambda$. Then

$$\operatorname{Irr}(H_k) = \{L_k^\lambda \mid \lambda \in \Lambda \text{ such that } \phi_k^\lambda \neq 0\}.$$

$H_k$ is semisimple if and only if $\phi_k^\lambda \neq 0$ and $L_k^\lambda = W_k^\lambda$ for all $\lambda \in \Lambda$.

Now let again $\mathcal{H} = \mathcal{H}_A(W)$ be the generic Iwahori–Hecke algebra.

**Aim:** General construction of a "cell datum" for $\mathcal{H}$. Start with:

> KAZHDAN–LUSZTIG basis of $\mathcal{H}$: $\{\mathbf{C}_w \mid w \in W\}$.

Write $v = u^{1/2}$ and $\tilde{T}_w = v^{-l(w)} T_w$ for all $w \in W$.

There is a unique ring involution $A \to A$, $a \mapsto \bar{a}$, such that $\bar{v} = v^{-1}$.

We can extend this to a ring involution $\mathcal{H} \to \mathcal{H}$, $h \mapsto \overline{h}$, such that

$$\overline{\sum_{w \in W} a_w \tilde{T}_w} = \sum_{w \in W} \bar{a}_w \tilde{T}_{w^{-1}}^{-1} \qquad (a_w \in A).$$

Then, for each $w \in W$, there exists a unique $\mathbf{C}_w \in \mathcal{H}$ such that

$$\overline{\mathbf{C}}_w = \mathbf{C}_w \qquad \text{and} \qquad \mathbf{C}_w \equiv (-1)^{l(w)} \tilde{T}_w \quad \mathrm{mod}\ \mathcal{H}_{>0},$$

where $\mathcal{H}_{>0} := \sum_{w \in W} A_{>0} \tilde{T}_w$ and $A_{>0} := v\mathbb{Z}[v]$.

Structure constants: Write

$$\mathbf{C}_x\mathbf{C}_y = \sum_{z\in W} h_{x,y,z}\,\mathbf{C}_z \qquad \text{where } h_{x,y,z} \in A.$$

G. Lusztig (1985): Define function $\mathbf{a}\colon W \to \mathbb{N}_0$ by

$$\mathbf{a}(z) = \min\{i \geqslant 0 \mid v^i h_{x,y,z} \in \mathbb{Z}[v]\ \forall x, y \in W\}.$$

Easy to see: $\mathbf{a}(1) = 0$, $\mathbf{a}(z) = \mathbf{a}(z^{-1})$ for all $z \in W$.

Given $x, y, z \in W$, set $\gamma_{x,y,z}$ = constant term of $v^{\mathbf{a}(z)} h_{x,y,z}$.

Let $J$ be a free abelian group with basis $\{t_w \mid w \in W\}$. Define

$$t_x \cdot t_y = \sum_{z\in W} \gamma_{x,y,z}\, t_z.$$

**Theorem.** G. Lusztig (1987)

$J$ is an associative ring with identity, such that $\mathbb{Q}W \cong \mathbb{Q} \otimes_{\mathbb{Z}} J$.

(Proof uses "positivity properties" arising from geometric interpretation of $\{\mathbf{C}_w\}$.)

Let $J_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} J$. Since $J_{\mathbb{Q}} \cong \mathbb{Q}W$, we can identify

$$\mathrm{Irr}(J_{\mathbb{Q}}) = \mathrm{Irr}_{\mathbb{Q}}(W) = \{E^\lambda \mid \lambda \in \Lambda\}.$$

For each $\lambda \in \Lambda$, there is a basis $M(\lambda)$ of $E^\lambda$ such that

$$t_w.\mathfrak{s}' = \sum_{\mathfrak{s} \in M(\lambda)} \underbrace{\rho^\lambda_{\mathfrak{s}\mathfrak{s}'}(t_w)}_{\in \mathbb{Z}}\, \mathfrak{s} \qquad \text{for any } \mathfrak{s}' \in M(\lambda).$$

**$J$ is a symmetric algebra $\rightsquigarrow$ Schur relations:**

$$\sum_{w \in W} \rho^\lambda_{\mathfrak{s}\mathfrak{t}}(t_w)\, \rho^\mu_{\mathfrak{u}\mathfrak{v}}(t_{w^{-1}}) = \begin{cases} f_\lambda & \text{if } \lambda = \mu,\ \mathfrak{s} = \mathfrak{v},\ \mathfrak{t} = \mathfrak{u}, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \neq f_\lambda \in \mathbb{N}$ for all $\lambda \in \Lambda$.

We say that a prime number $p$ is bad if $p \mid f_\lambda$ for some $\lambda \in \Lambda$.

| | | | |
|---|---|---|---|
| $A_n$: | no bad $p$; | $G_2, F_4, E_6, E_7$: | $p = 2, 3$; |
| $B_n, C_n, D_n$: | $p = 2$; | $E_8$: | $p = 2, 3, 5$. |

Recall that $\mathcal{H}$ is defined over $A = R[u^{1/2}, u^{-1/2}]$. Now choose $R \subseteq \mathbb{C}$ such that all bad primes are invertible in $R$. Consider the data:

- $\operatorname{Irr}(J_{\mathbb{Q}}) = \operatorname{Irr}_{\mathbb{Q}}(W) = \{E^\lambda \mid \lambda \in \Lambda\}$, $\qquad M(\lambda)$ basis of $E^\lambda$,
- $\left(\rho^\lambda_{\mathfrak{st}}(t_w)\right)$ $\qquad$ integral representations of $J$.

Furthermore, $\exists\, B^\lambda = \left(\beta^\lambda_{\mathfrak{st}}\right)$ such that $\det B^\lambda \not\equiv 0 \bmod p$ (for $p$ not bad), $(B^\lambda)^{\mathrm{tr}} = B^\lambda$, and $B^\lambda \cdot \rho^\lambda(t_{w^{-1}}) = \rho^\lambda(t_w)^{\mathrm{tr}} \cdot B^\lambda \;\; \forall w \in W$.

**Theorem.** M.G. (2006)

For any $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, define

$$\boxed{C^\lambda_{\mathfrak{s},\mathfrak{t}} = \sum_{w \in W} \sum_{\mathfrak{u} \in M(\lambda)} \beta^\lambda_{\mathfrak{tu}}\, \rho^\lambda_{\mathfrak{us}}\,(t_{w^{-1}})\, \mathbf{C}_w.}$$

Then $\{\, C^\lambda_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{t} \in M(\lambda) \,\}$ is a "cellular basis" arising from a "cell datum" for $\mathcal{H}$ in the sense of Graham–Lehrer.

The anti-involution $*\colon \mathcal{H} \to \mathcal{H}$ is given $T_w^* = T_{w^{-1}}$.

The required partial order on $\Lambda$ is defined as follows:

For $i \geqslant 0$, let $J_{\mathbb{Q}}^i = \langle\, t_w \mid \mathbf{a}(w) = i \,\rangle_{\mathbb{Q}}$ (subspace). Then

$$J_{\mathbb{Q}}^i \text{ is a two-sided ideal of } J_{\mathbb{Q}} \quad \text{and} \quad J_{\mathbb{Q}} = \bigoplus_{i \geqslant 0} J_{\mathbb{Q}}^i.$$

Hence, given $\lambda \in \Lambda$, we have $J_{\mathbb{Q}}^i.E^\lambda \neq 0$ for a unique $i \in \mathbb{N}_0$; write $\mathbf{a}_\lambda = i$. Thus, we obtain a function

$$\mathrm{Irr}_{\mathbb{Q}}(W) \to \mathbb{N}_0, \quad E^\lambda \mapsto \mathbf{a}_\lambda \qquad \text{(G. Lusztig, 1987).}$$

Using previous work by many authors on so-called “generic degrees” (most notably, P. N. Hoefsmit, 1974), the invariants $\mathbf{a}_\lambda$ are explicitly known in all cases.

Define: $\lambda \leqslant \mu$ iff $\lambda = \mu$ or $\mathbf{a}_\mu < \mathbf{a}_\lambda$.

Let $W = W(A_{n-1}) = \mathfrak{S}_n$. Special feature: $f_\lambda = 1$ for all $\lambda$.

Expression for $C^\lambda_{\mathfrak{s},\mathfrak{t}}$ reduces to one term. Hence

$$\{\mathbf{C}_w \mid w \in W\} \text{ is a cellular basis.}$$

This example already appeared in the article of Graham and Lehrer.

Another cellular basis was constructed by G. E. Murphy (1992), by an ingenious combinatorial argument (and even before "cell data" were introduced). For the relation between the two cellular bases, see

- T. McDonough and C. Pallikaros (2005): Cell modules (=Kazhdan–Lusztig cell modules in this case) are canonically isomorphic to Dipper–James Specht modules.
- M.G., "Kazhdan–Lusztig cells and the Murphy basis" (2006).

Let $W = W(B_2)$, with generators $S = \{s_1, s_2\}$ such that $(s_1 s_2)^4 = 1$.

$$\mathrm{Irr}_{\mathbb{Q}}(W) = \{\mathbf{1}, \varepsilon_1, \varepsilon_2, \varepsilon, r\}$$

where $\mathbf{1}$ =unit, $\varepsilon$ =sign, $\dim \varepsilon_1 = \dim \varepsilon_2 = 1$, $\dim r = 2$.

$$\mathbf{a}_{\mathbf{1}} = 0,\ \mathbf{a}_{\varepsilon_1} = \mathbf{a}_{\varepsilon_2} = \mathbf{a}_r = 1 \text{ and } \mathbf{a}_\varepsilon = 4; \qquad f_\lambda = 1 \text{ or } 2.$$

Cellular basis:

$$\begin{aligned}
C^{\mathbf{1}}_{1,1} &= \mathbf{C}_1, & C^{r}_{1,1} &= \mathbf{C}_{s_1} + \mathbf{C}_{s_1 s_2 s_1},\\
C^{\varepsilon}_{1,1} &= \mathbf{C}_{s_1 s_2 s_1 s_2}, & C^{r}_{1,2} &= -2\mathbf{C}_{s_1 s_2},\\
C^{\varepsilon_1}_{1,1} &= \mathbf{C}_{s_2} - \mathbf{C}_{s_2 s_1 s_2}, & C^{r}_{2,1} &= -2\mathbf{C}_{s_2 s_1},\\
C^{\varepsilon_2}_{1,1} &= \mathbf{C}_{s_1} - \mathbf{C}_{s_1 s_2 s_1}, & C^{r}_{2,2} &= 2\mathbf{C}_{s_2} + 2\mathbf{C}_{s_2 s_1 s_2}.
\end{aligned}$$

**Final remark:** There is also a version of the Theorem for Hecke algebras with unequal parameters, but then the proofs rely on the validity of the conjectures in Lusztig's book (2003).

# Part II

Recall: $W$ finite Weyl group, with generating set $S$.

$R \subseteq \mathbb{C}$ (smallest) subring in which all bad primes for $W$ are invertible.

$\mathcal{H} = \mathcal{H}_A(W)$ generic Iwahori–Hecke algebra over $A = R[u^{1/2}, u^{-1/2}]$.

Kazhdan–Lusztig theory $\rightsquigarrow$

$$\text{“cellular basis”} \qquad \{C^\lambda_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \Lambda, \mathfrak{s}, \mathfrak{t} \in M(\lambda)\}.$$

Cell modules $W^\lambda$ (for $\lambda \in \Lambda$) with invariant bilinear form $\phi^\lambda$.

Consider a specialisation $\theta\colon A \to k$ ($k$ field), $u \mapsto \xi$. Write

$$\mathcal{H}_\xi = k \otimes_A \mathcal{H} \text{ and } W^\lambda_\xi = k \otimes_A W^\lambda, \text{ with induced form } \phi^\lambda_\xi.$$

Set $L^\lambda_\xi = W^\lambda_\xi / \text{rad}(\phi^\lambda_\xi)$. Then

$$\text{Irr}(\mathcal{H}_\xi) = \{L^\lambda_\xi \mid \lambda \in \Lambda^\circ_\xi\} \quad \text{where} \quad \Lambda^\circ_\xi = \{\lambda \in \Lambda \mid \phi^\lambda_\xi \neq 0\}.$$

$\mathcal{H}_\xi$ is semisimple if and only if $\Lambda^\circ_\xi = \Lambda$ and $L^\lambda_\xi = W^\lambda_\xi$ for all $\lambda \in \Lambda$.

**General major open problem:**

Find explicit formulas for $\dim L_\xi^\lambda$ or, equivalently, for $\dim \operatorname{rad}(\phi_\xi^\lambda)$.

In particular, determine the subset $\Lambda_\xi^\circ \subseteq \Lambda$.

Furthermore, determine the decomposition matrix

$$D_\xi = \Big([W_\xi^\lambda : L_\xi^\mu]\Big)_{\lambda\in\Lambda,\ \mu\in\Lambda_\xi^\circ}$$

As before, we set

$$e = \min\{i \geqslant 2 \mid 1+\xi+\xi^2+\cdots+\xi^{i-1}=0\}.$$

($e=\infty$ if no such $i$ exists.)

Recall: Let $W = W(A_{n-1}) \cong \mathfrak{S}_n$. Trivial case: If $e > n$, then $\mathcal{H}_\xi = H_n(k,\xi)$ is semisimple and $L_\xi^\lambda = W_\xi^\lambda$ for all $\lambda \vdash n$.

What is the analogous condition for $W$ in general?

Poincaré polynomial $$P_W = \sum_{w \in W} u^{l(w)} = \prod_{i=1}^{|S|} \frac{u^{d_i} - 1}{u - 1}$$

where $d_1, \ldots, d_{|S|}$ are the *degrees* of $W$; we have $|W| = d_1 \cdots d_{|S|}$.

| Type | degrees $d_i$ |
|---|---|
| $A_{n-1}$ | $2, 3, 4, \ldots, n$ |
| $B_n, C_n$ | $2, 4, 6, \ldots, 2n$ |
| $D_n$ | $2, 4, 6, \ldots, 2(n-1), n$ |

| Type | degrees $d_i$ |
|---|---|
| $G_2$ | $2, 6$ |
| $F_4$ | $2, 6, 8, 12$ |
| $E_6$ | $2, 5, 6, 8, 9, 12$ |
| $E_7$ | $2, 6, 8, 10, 12, 14, 18$ |
| $E_8$ | $2, 8, 12, 14, 18, 20, 24, 30$ |

**Proposition.** A. Gyoja and K. Uno (1989)

$\mathcal{H}_\xi$ is semisimple if and only if $P_W(\xi) \neq 0$.

**Corollary.** If $e = \infty$ or $e$ does not divide any degree of $W$, then

$\mathcal{H}_\xi$ is semisimple, $\Lambda_\xi^\circ = \Lambda$ and $L_\xi^\lambda = W_\xi^\lambda$ for all $\lambda \in \Lambda$.

Given $\theta\colon A \to k$, $u \mapsto \xi$, define $e \geqslant 2$ as above.

We assume from now on that $e < \infty$ and let $\zeta_e = \sqrt[e]{1} \in \mathbb{C}$.

We will want to compare the representations of $\mathcal{H}_\xi$ and $\mathcal{H}_{\zeta_e}$.

Note: $\Phi_e(\xi) = 0$ where $\Phi_e \in \mathbb{Z}[u]$ is the $e$th cyclotomic polynomial.

So, $\exists$ ring homomorphism $R[\zeta_e] \to k$, $a \mapsto \bar{a}$, such that $\theta$ is the composition of the specialisation $A \to R[\zeta_e]$, $u \mapsto \zeta_e$, with $a \mapsto \bar{a}$.

Let $P^\lambda$ be the Gram matrix of $\phi^\lambda$. Then $\quad P^\lambda_\xi = \overline{P^\lambda_{\zeta_e}}$

Hence, if $P^\lambda_{\zeta_e} = 0$, then $P^\lambda_\xi = 0$. Similarly, $\operatorname{rank}(P^\lambda_\xi) \leqslant \operatorname{rank}(P^\lambda_{\zeta_e})$.

So we obtain:

**Lemma.**

$$\Lambda^\circ_\xi \subseteq \Lambda^\circ_{\zeta_e} \quad \text{and} \quad \dim L^\mu_\xi \leqslant \dim L^\mu_{\zeta_e} \quad \text{for all } \mu \in \Lambda^\circ_\xi.$$

Recall: Let $W = W(A_{n-1}) \cong \mathfrak{S}_n$. Then James' conjectures involves the hypothesis that $e\ell > n$.

What is the the analogous condition for $W$ in general?

**Theorem.** M. G. and R. Rouquier (1997)

Assume that $e < \infty$, $\operatorname{char}(k) = \ell > 0$ and that

(∗) $e\ell$ does not divide any degree of $W$.

Consider the algebra $\mathcal{H}_{\zeta_e}$ where $\zeta_e = \sqrt[e]{1} \in \mathbb{C}$. Then

$$|\operatorname{Irr}(\mathcal{H}_\xi)| = |\operatorname{Irr}(\mathcal{H}_{\zeta_e})|.$$

**Corollary.** N. Jacon (2004)

Under the above assumptions, we have $\Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ$.

## General version of James' Conjecture

Assume that $e < \infty$, $\mathrm{char}(k) = \ell > 0$ and that

$$(*) \qquad \underline{e\ell \text{ does not divide any degree of } W}.$$

Then the decomposition matrix $D_\xi$ only depends on $e$.

More precisely, consider the algebra $\mathcal{H}_{\zeta_e}$ where $\zeta_e = \sqrt[e]{1} \in \mathbb{C}$. Then

$$[W_\xi^\lambda : L_\xi^\mu] = [W_{\zeta_e}^\lambda : L_{\zeta_e}^\mu] \qquad \forall\, \lambda \in \Lambda,\;\; \mu \in \Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ.$$

## Remark.

In the formulation of M. G. (1990), condition $(*)$ was replaced by the stronger condition that $\ell$ does not divide $|W|$.

It was shown by a general argument that, if $\ell \nmid |W|$, $\xi^{\ell-1} = 1$ and if $\zeta_e$ is a simple root of $P_W$, then James' conjecture holds.

Recall from the theory of cellular algebras that there is a partial order $\leqslant$ on $\Lambda$. If we choose a total order which refines $\leqslant$, then the matrices

$$D_\xi^\circ = \Big([W_\xi^\lambda : L_\xi^\mu]\Big)_{\lambda,\mu\in\Lambda_\xi^\circ} \qquad \text{and} \qquad D_{\zeta_e}^\circ = \Big([W_{\zeta_e}^\lambda : L_{\zeta_e}^\mu]\Big)_{\lambda,\mu\in\Lambda_{\zeta_e}^\circ}$$

are square, lower triangular, with 1 on the diagonal.

Now assume that the hypotheses of James' conjecture are satisfied. Then $\Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ$ and a general factorisation result on decomposition matrices due to M. G. and R. Rouquier (1997) implies that

$$D_\xi^\circ = D_{\zeta_e}^\circ \cdot A_\xi$$

where $A_\xi$ is a triangular matrix with non-negative integer coefficients and 1 on the diagonal. The matrix $A_\xi$ already appeared in James' work on type $A_{n-1}$ where he called it adjustment matrix.

**Corollary.** (Alternative version of James' Conjecture)

Assume that $e < \infty$, $\mathrm{char}(k) = \ell > 0$ and that $e\ell$ does not divide any degree of $W$. Then the following are equivalent:

1. James' conjecture holds;
2. $A_\xi$ is the identity matrix;
3. $\dim L_\xi^\mu = \dim L_{\zeta_e}^\mu$ for all $\mu \in \Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ$.
4. $\dim \mathrm{rad}(\phi_\xi^\lambda) = \dim \mathrm{rad}(\phi_{\zeta_e}^\lambda)$ for all $\lambda \in \Lambda$.

Condition 4 shows that James' conjecture can be effictively verified once we know the Gram matrices of the bilinear forms $\phi^\lambda$ and determine their ranks for various specialisations.

## So, how do we actually compute the Gram matrix of $\phi^\lambda$ ?

Recall that $W^\lambda$ has basis $\{C_\mathfrak{s} \mid \mathfrak{s} \in M(\lambda)\}$ and $\phi^\lambda$ is determined by

$$C^\lambda_{\mathfrak{s}',\mathfrak{s}} C^\lambda_{\mathfrak{t},\mathfrak{t}'} = \phi^\lambda(C_\mathfrak{s}, C_\mathfrak{t})\, C^\lambda_{\mathfrak{s}',\mathfrak{t}'} \quad + \quad \text{“lower terms”}.$$

Thus, the coefficients of the Gram matrix of $\phi^\lambda$ are structure constants for $\mathcal{H}$ and, hence, can be computed in principle. Recall:

$$C^\lambda_{\mathfrak{s},\mathfrak{t}} = \text{ certain linear combination of } \mathbf{C}_w\text{'s.}$$

So, first of all, we need to be able to work out structure constants for the Kazhdan–Lusztig basis (the $h_{x,y,z}$'s). Also possible in principle.

However, all this may be feasable in type $G_2$ or even $F_4$, but it is totally unrealistic to carry out such computations for type $E_8$.

We will come back to these computational issues a little later.

**Corollary.** The conclusion of James' conjecture holds for all $\ell \gg 0$.

**Proof.** Fix $e \geqslant 2$ and $\lambda \in \Lambda$. Let $P^\lambda$ be the Gram matrix of $\phi^\lambda$. Then

$$P_\xi^\lambda = \overline{P_{\zeta_e}^\lambda}$$

where $R[\zeta_e] \to k$, $a \mapsto \bar{a}$, is a ring homomorphism such that

$\theta\colon A \to k$, $u \mapsto \xi$, is obtained by first specialising $A \to R[\zeta_e]$,

$u \mapsto \zeta_e$, and then applying $a \mapsto \bar{a}$.

We already noticed that $\operatorname{rank}(P_\xi^\lambda) \leqslant \operatorname{rank}(P_{\zeta_e}^\lambda)$.

On the other hand, let $P'$ be any square submatrix of $P^\lambda$. Then:

$$\det(P'_{\zeta_e}) \neq 0 \quad \Rightarrow \quad \det(P'_\xi) = \det(\overline{P_{\zeta_e}^\lambda}) = \overline{\det(P_{\zeta_e}^\lambda)} \neq 0 \text{ for } \ell \gg 0.$$

So $\operatorname{rank}(P_\xi^\lambda) = \operatorname{rank}(P_{\zeta_e}^\lambda)$ for $\ell \gg 0$, i.e., Condition 4 holds for $\ell \gg 0$.

**From now on:** Joint work with J. Müller

**Aim:** Systematic verification of James' conjecture for $W$ of exceptional type, including $E_8$ !

Previous work:

type $G_2$: easy (folklore);

type $F_4$: M. G. and K. Lux (1991); type $E_6$: M. G. (1993).

(Methods were rather ad hoc and cannot be made to work for $E_8$.)

New attempt:

1. Determine $\Lambda^\circ_{\zeta_e}$ and compute $[W^\lambda_{\zeta_e} : L^\mu_{\zeta_e}]$ for all $e \geqslant 2$.
2. Determine the Gram matrix $P^\lambda$ of $\phi^\lambda$ for $\lambda \in \Lambda^\circ_{\zeta_e}$.
3. Find bound on $\ell$ such that $\operatorname{rank}(P^\lambda_{\zeta_e}) = \operatorname{rank}(P^\lambda_{\xi}) = \operatorname{rank}(\overline{P^\lambda_{\zeta_e}})$.

Let $K = \mathrm{Quot}(A)$ and $\mathcal{H}_K = K \otimes_A \mathcal{H}$, a split semisimple algebra.

By the theory of cellular algebras:

$$\mathrm{Irr}(\mathcal{H}_K) = \{W_K^\lambda \mid \lambda \in \Lambda\}.$$

A. Gyoja (1984): Every irreducible representation of $\mathcal{H}_K$ can described by a *W-graph*, a combinatorial datum introduced by G. Lusztig and D. Kazhdan (1979).

H. Naruse (1998) explicitly determined $W$-graphs for types $F_4$, $E_6$;
R. Howlett–Y. Yin, R. Howlett ($\sim$ 2003) did $E_7$, $E_8$.
J. Michel ($\sim$ 2004) implemented these $W$-graphs in GAP.

Thus, for each $\lambda \in \Lambda$, we have a collection of matrices $\rho^\lambda(T_s)$ ($s \in S$) which afford $W_K^\lambda$; the coefficients of $\rho^\lambda(T_s)$ are in $A$.

Now specialise $u \mapsto \zeta_e$ and obtain a representation $\rho^\lambda_{\zeta_e}$ of $\mathcal{H}_{\zeta_e}$.

Use R. Parker's **MeatAxe** programs (developped in the 1980's with the aim of constructing matrix representations of sporadic simple groups) to decompose each $\rho^\lambda_{\zeta_e}$ into its irreducible constituents.

Thus, we obtain:

- $\mathrm{Irr}(\mathcal{H}_{\zeta_e}) = \{M_1, \ldots, M_r\}$ where $r = |\mathrm{Irr}(\mathcal{H}_{\zeta_e})|$;
- composition multiplicities $[W^\lambda_{\zeta_e} : M_i]$ for $\lambda \in \Lambda$ and $1 \leqslant i \leqslant r$.

The general theory of cellular algebras implies:

**Lemma.** Suppose $M \in \mathrm{Irr}(\mathcal{H}_{\zeta_e})$. Then the unique $\mu \in \Lambda^\circ_{\zeta_e}$ such that $M = L^\mu_{\zeta_e}$ is determined by the conditions that $[W^\mu_{\zeta_e} : M] = 1$ and

$$\mathbf{a}_\mu \leqslant \mathbf{a}_\lambda \quad \text{for all } \lambda \in \Lambda \text{ such that } [W^\lambda_{\zeta_e} : M] \neq 0.$$

$\mathcal{H} = \mathcal{H}(E_6)$

$|\Lambda| = 25$

$e = 2$

$u \mapsto \zeta_2 = -1$

Dec. mat.:

M.G. (1993)

$|\operatorname{Irr}(\mathcal{H}_{-1})| = 8$

| $\lambda$ | $[W_{-1}^{\lambda} : M]$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_p$ | 1 | . | . | . | . | . | . | . |
| $1'_p$ | 1 | . | . | . | . | . | . | . |
| $6_p$ | . | 1 | . | . | . | . | . | . |
| $6'_p$ | . | 1 | . | . | . | . | . | . |
| $10_s$ | . | . | 1 | . | . | . | . | . |
| $15_q$ | 1 | . | . | 1 | . | . | . | . |
| $15_p$ | 1 | . | . | . | 1 | . | . | . |
| $15'_q$ | 1 | . | . | 1 | . | . | . | . |
| $15'_p$ | 1 | . | . | . | 1 | . | . | . |
| $20_p$ | . | 1 | . | . | 1 | . | . | . |
| $20_s$ | . | 1 | . | 1 | . | . | . | . |
| $20'_p$ | . | 1 | . | . | 1 | . | . | . |
| $24_p$ | . | . | 1 | . | 1 | . | . | . |
| $24'_p$ | . | . | 1 | . | 1 | . | . | . |
| $30_p$ | . | 1 | 1 | . | 1 | . | . | . |
| $30'_p$ | . | 1 | 1 | . | 1 | . | . | . |
| $60_p$ | . | . | . | . | 1 | 1 | . | . |
| $60_s$ | . | . | . | 1 | . | 1 | . | . |
| $60'_p$ | . | . | . | . | 1 | 1 | . | . |
| $64_p$ | . | . | . | . | . | . | 1 | . |
| $64'_p$ | . | . | . | . | . | . | 1 | . |
| $80_s$ | . | . | . | . | . | . | . | 1 |
| $81_p$ | 1 | 1 | . | 1 | 1 | 1 | . | . |
| $81'_p$ | 1 | 1 | . | 1 | 1 | 1 | . | . |
| $90_s$ | . | 1 | 1 | . | 2 | 1 | . | . |

reorder

using $a_\lambda$

$\rightsquigarrow \Lambda^\circ_{-1}$

(in blue)

| $\lambda$ | $a_\lambda$ | $[W_{-1}^{\lambda} : M]$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **$1_p$** | **0** | **1** | . | . | . | . | . | . | . |
| **$6_p$** | **1** | . | **1** | . | . | . | . | . | . |
| **$20_p$** | **2** | . | **1** | **1** | . | . | . | . | . |
| $15_p$ | 3 | 1 | . | 1 | . | . | . | . | . |
| **$15_q$** | **3** | **1** | . | . | **1** | . | . | . | . |
| **$30_p$** | **3** | . | **1** | **1** | . | **1** | . | . | . |
| **$64_p$** | **4** | . | . | . | . | . | **1** | . | . |
| **$60_p$** | **5** | . | . | **1** | . | . | . | **1** | . |
| $24_p$ | 6 | . | . | 1 | . | 1 | . | . | . |
| $81_p$ | 6 | 1 | 1 | 1 | 1 | . | . | 1 | . |
| $10_s$ | 7 | . | . | . | . | 1 | . | . | . |
| $20_s$ | 7 | . | 1 | . | 1 | . | . | . | . |
| $60_s$ | 7 | . | . | . | 1 | . | . | 1 | . |
| **$80_s$** | **7** | . | . | . | . | . | . | . | **1** |
| $90_s$ | 7 | . | 1 | 2 | . | 1 | . | 1 | . |
| $81'_p$ | 10 | 1 | 1 | 1 | 1 | . | . | 1 | . |
| $60'_p$ | 11 | . | . | 1 | . | . | . | 1 | . |
| $24'_p$ | 12 | . | . | 1 | . | 1 | . | . | . |
| $64'_p$ | 13 | . | . | . | . | . | 1 | . | . |
| $15'_p$ | 15 | 1 | . | 1 | . | . | . | . | . |
| $15'_q$ | 15 | 1 | . | . | 1 | . | . | . | . |
| $30'_p$ | 15 | . | 1 | 1 | . | 1 | . | . | . |
| $20'_p$ | 20 | . | 1 | 1 | . | . | . | . | . |
| $6'_p$ | 25 | . | 1 | . | . | . | . | . | . |
| $1'_p$ | 36 | 1 | . | . | . | . | . | . | . |

## Another try to find the Gram matrix of $\phi^\lambda$.

Consider matrix representation (coming from $W$-graph):

$$\rho^\lambda \colon \mathcal{H} \to M_{d_\lambda}(A), \qquad \text{where} \qquad d_\lambda = |M(\lambda)|.$$

Let $P^\lambda$ be the Gram matrix of $\phi^\lambda$. Then the $\mathcal{H}$-invariance of $\phi^\lambda$ is equivalent to:

$$(*) \qquad P^\lambda \cdot \rho^\lambda(T_s) = \rho^\lambda(T_s)^{\text{tr}} \cdot P^\lambda \quad \forall\, s \in S.$$

Since $\rho^\lambda$ defines an irreducible representation of $\mathcal{H}_K$, condition $(*)$ uniquely determines $P^\lambda$ up to scalar multiples (Schur's Lemma).

$(*)$ is a system of $|S|d_\lambda^2$ linear equations for the $d_\lambda^2$ coefficients of $P^\lambda$.

This seems feasable for $W$ of type $G_2$, $F_4$, $E_6$ (where $d_\lambda \leqslant 90$) but, again, it is totally unrealistic in type $E_8$ (where $\exists\, d_\lambda = 7168$).

Now note: Given $\rho^\lambda \colon \mathcal{H} \to M_{d_\lambda}(A)$, the map

$$\hat{\rho}^\lambda \colon \mathcal{H} \to M_{d_\lambda}(A), \qquad T_w \mapsto \rho^\lambda(T_{w^{-1}})^{\mathrm{tr}}$$

also is a representation. Thus, $(*)$ means that $P^\lambda$ is the matrix of an $\mathcal{H}$-module homomorphism between $\rho^\lambda$ and $\hat{\rho}^\lambda$.

**Theorem.** C. W. Curtis and C. T. Benson (1972)

Every $\rho^\lambda$ is of parabolic type, that is, $\exists\, J \subseteq S$ such that

$$\dim_K \Big( \bigcap_{s \in J} \ker\big(\rho^\lambda(T_s) - u\,\mathrm{Id}_{d_\lambda}\big)\Big) = 1.$$

Let $v_1 \neq 0$ be a vector in the above intersection.

Let $\hat{v}_1 \neq 0$ be in the analogous intersection with respect to $\hat{\rho}^\lambda$.

Note: By the above theorem, $v_1, \hat{v}_1$ are unique up to scalar.

So condition $(*)$ implies that $\hat{v}_1 = \alpha\, P^\lambda v_1$ for some scalar $\alpha \neq 0$.

Since $\rho^\lambda$ irreducible (over $K$), we have $K^{d_\lambda} = \langle \rho^\lambda(T_w)v_1 \mid w \in W\rangle_K$.

Applying repeatedly $T_s$ ($s \in S$), find $w_2, \ldots, w_{d_\lambda} \in W$ such that

$$\left.\begin{aligned} & v_1 \\ & v_2 = \rho^\lambda(T_{w_2})v_1 \\ & \qquad\vdots \\ & v_{d_\lambda} = \rho^\lambda(T_{w_{d_\lambda}})v_1 \end{aligned}\right\} \quad \text{form a basis of } K^{d_\lambda}.$$

Then

$$\left.\begin{aligned} & \hat{v}_1 \\ & \hat{v}_2 = \hat{\rho}^\lambda(T_{w_2})\hat{v}_1 \\ & \qquad\vdots \\ & \hat{v}_{d_\lambda} = \hat{\rho}^\lambda(T_{w_{d_\lambda}})\hat{v}_1 \end{aligned}\right\} \quad \text{also form a basis of } K^{d_\lambda},$$

and we have $\hat{v}_i = \alpha P^\lambda v_i$ for all $i$. Thus, can compute $P^\lambda$.

To perform these computations, all we need is:

- doing Gaussian eliminations;
- multiplying a matrix by a vector.

Actually, we cannot do this over $K = \mathrm{Quot}(A)$ (coefficient overflow!) Instead, specialize $u$ to various natural numbers, then reduce modulo various primes. Using Chinese remainder + interpolation, recover $P^\lambda$. (Note, given a candidate matrix $P^\lambda$, we can just check if $(*)$ holds.)

**Theorem/Fact.** M. Geck and J. Müller (2007)

The general version of James' conjecture is true for $W$ of exceptional type; the decomposition numbers $[W^\lambda_{\zeta_e} : L^\mu_{\zeta_e}]$ are explicitly known in the form of tables.