

On the computation of character values for finite Chevalley groups of exceptional type

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Abstract: We discuss various computational issues around the problem of determining the character values of finite Chevalley groups, in the framework provided by Lusztig’s theory of character sheaves. Some of the remaining open questions (concerning certain roots of unity) for the cuspidal unipotent character sheaves of groups of exceptional type are resolved.

Keywords: Groups of Lie type, Deligne–Lusztig characters, character sheaves.

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1. Introduction

Let p be a prime and $k = \overline{\mathbb{F}}_p$ be an algebraic closure of the field with p elements. Let \mathbf{G} be a connected reductive algebraic group over k and assume

arXiv: [2105.00722](https://arxiv.org/abs/2105.00722)

Received July 5, 2021.

2010 Mathematics Subject Classification: Primary 20C33; secondary 20G402.

that \mathbf{G} is defined over the finite subfield $\mathbb{F}_q \subseteq k$, where q is a power of p . Let $F: \mathbf{G} \rightarrow \mathbf{G}$ be the corresponding Frobenius map. Then the group of rational points $\mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$ is called a “finite group of Lie type”. (For the basic theory of these groups, see [4], [10], [21].) We are concerned with the problem of computing the values of the irreducible characters of \mathbf{G}^F . The work of Lusztig [33], [41], [45] has led to a general program for solving this problem. In this framework, one seeks to establish certain identities of class functions on \mathbf{G}^F of the form $R_x = \zeta \chi_A$, where R_x denotes an “almost character” (that is, an explicitly defined linear combination of the irreducible characters of \mathbf{G}^F) and χ_A denotes the characteristic function of a suitable F -invariant “character sheaf” A on \mathbf{G} ; here, ζ is an algebraic number of absolute value 1. This program has been successfully carried out in many cases (see [21, §2.7] for a survey), but not in complete generality.

This paper is part of an ongoing project (involving various authors) to complete the program of establishing identities $R_x = \zeta \chi_A$ as above including the explicit determination of the scalars ζ . We shall solve this problem here in a number of previously open cases for \mathbf{G} simple of exceptional type.

The above identities $R_x = \zeta \chi_A$ take a particularly striking form when A is a cuspidal character sheaf and \mathbf{G} is a simple algebraic group, and this is our main focus here. In that case, the set $\{g \in \mathbf{G}^F \mid \chi_A(g) \neq 0\}$ is contained in a single F -stable conjugacy class Σ of \mathbf{G} ; furthermore, the values of χ_A are determined by the choice of an element $g_1 \in \Sigma^F$ and a certain irreducible character ψ of the finite group of components $A_{\mathbf{G}}(g_1) = C_{\mathbf{G}}(g_1)/C_{\mathbf{G}}^{\circ}(g_1)$. By [41, 0.4], the general case can be reduced to the “cuspidal” case assuming that the cohomological induction functor $R_{\mathbf{L}}^{\mathbf{G}}$ (see [32], [10, §9.1]) is explicitly known, where $\mathbf{L} \subseteq \mathbf{G}$ is any F -stable Levi subgroup of a not necessarily F -stable parabolic subgroup of \mathbf{G} .

A crucial ingredient in this whole program is the problem of identifying “good” choices for $g_1 \in \Sigma^F$ as above. If Σ is a unipotent class, then one can use the concept of “split” elements; see Beynon–Spaltenstein [1] and Shoji [55]. In general, there are a few rare cases where one can single out a representative $g_1 \in \Sigma^F$ simply by looking at the order or the structure of the centralisers. At the other extreme, all $g_1 \in \Sigma^F$ may have the same centraliser order. In such cases, we use the following techniques: 1) Steinberg’s cross-section [59] for regular elements or, more generally, Lusztig’s “ C -small” classes [43]; 2) rationality properties of characters; 3) powermaps and congruence conditions for character values.

When dealing with concrete examples, we will usually assume that \mathbf{G} is simple, simply connected in order to facilitate the use of the above techniques and computations with semisimple elements. Our aim is to achieve something

close to the famous Cambridge ATLAS [6], where no explicit representatives of conjugacy classes are given, but the classes can be almost uniquely identified by some formal properties, like 2) or 3). For many applications of character theory (for examples, see [21, App. A.10]) this is entirely sufficient. In Section 4, we develop some techniques that will help us identifying such “good” choices for $g_1 \in \Sigma^F$. (Note, however, that there does not yet seem to be a universally applicable definition of what “good” should mean.)

But first we need to address another essential point: the explicit evaluation of the Deligne–Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, where \mathbf{T} is any F -stable maximal torus of \mathbf{G} and $\theta \in \text{Irr}(\mathbf{T}^F)$. There is a character formula in [7] which reduces that problem to the computation of Green functions. The formula involves some technical issues of a purely group-theoretical nature. It will be known to the experts how to deal with this, but the details are not readily available so we include them here in Sections 2 and 3 (following, and slightly revising Lübeck [30]). We hope that this will be useful as a reference in other contexts as well.

Finally, in Sections 5–7, we explicitly deal with cuspidal character sheaves in groups of types F_4 , E_6 , E_7 . Much of this is inspired by Lusztig [38] (values of characters on unipotent elements) and Shoji [52] (values of unipotent characters for classical groups); an additional complication here is that unipotent characters of exceptional groups may have non-rational values. We heavily rely on computer calculations, where we use Michel’s extremely powerful version of CHEVIE [47]. In addition to the general functions concerning Weyl groups, reflection subgroups and their character tables, there are programs in [47] for producing information about the unipotent characters of \mathbf{G}^F (degrees, Fourier matrices etc.), and for computing (generalised) Green functions, which turn out to be particularly helpful for our purposes here. Combined with previous work by a number of authors (for precise references see Sections 5–7), we can now state:

Let \mathbf{G} be simple of type G_2 , F_4 , E_6 or E_7 . Then the scalars ζ in the identities $R_x = \zeta \chi_A$ for cuspidal unipotent character sheaves A are explicitly known. In all cases considered, there is a “good” choice of $g_1 \in \Sigma^F$ such that $\zeta = 1$.

As far as simple groups of exceptional type are concerned, what remains to be done is to deal with a number of cuspidal character sheaves for \mathbf{G} of type E_8 (which are all unipotent) and with the non-unipotent cuspidal character sheaves for \mathbf{G} of type E_6 and E_7 . For type E_8 , see [27] (plus work in progress); the remaining cases for E_6 , E_7 (which only occur when \mathbf{G} is simply connected with a non-trivial center), will be considered in [22].

1.1. Notation and conventions. The set of (complex) irreducible characters of a finite group Γ is denoted by $\text{Irr}(\Gamma)$. We work over a fixed subfield $\mathbb{K} \subseteq \mathbb{C}$, which is algebraic over \mathbb{Q} , invariant under complex conjugation and “large enough”, that is, \mathbb{K} contains sufficiently many roots of unity and \mathbb{K} is a splitting field for Γ and all its subgroups. Thus, $\chi(g) \in \mathbb{K}$ for all $\chi \in \text{Irr}(\Gamma)$ and $g \in \Gamma$. When required, we will assume chosen an embedding of \mathbb{K} into $\overline{\mathbb{Q}_l}$, where \mathbb{Q}_l is the field of l -adic numbers for some prime l . If $\alpha: \Gamma \rightarrow \Gamma$ is a group automorphism, we say that $g_1, g_2 \in \Gamma$ are α -conjugate if there exists some $g \in \Gamma$ such that $g_2 = g^{-1}g_1\alpha(g)$.

2. Fusion of F -stable maximal tori

We keep the general notation from the introduction. Let \mathbf{T}_0 be a maximally split torus of \mathbf{G} , that is, \mathbf{T}_0 is an F -stable maximal torus contained in an F -stable Borel subgroup $\mathbf{B} \subseteq \mathbf{G}$. Let Φ be the root system of \mathbf{G} with respect to \mathbf{T}_0 , and let $\Phi^+ \subseteq \Phi$ be the set of positive roots determined by \mathbf{B} . Let $\mathbf{W} := N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ be the Weyl group of \mathbf{T}_0 and $\ell: \mathbf{W} \rightarrow \mathbb{Z}_{\geq 0}$ be the length function. We have $\mathbf{W} = \langle w_\alpha \mid \alpha \in \Phi \rangle$ where $w_\alpha \in \mathbf{W}$ denotes the reflection with root α . We have $\mathbf{G} = \langle \mathbf{T}_0, \mathbf{U}_\alpha \ (\alpha \in \Phi) \rangle$ where $\mathbf{U}_\alpha \subseteq \mathbf{G}$ denotes the root subgroup corresponding to α . The Frobenius map F induces a permutation $\alpha \mapsto \alpha^\dagger$ of Φ such that $F(\mathbf{U}_\alpha) = \mathbf{U}_{\alpha^\dagger}$ for all $\alpha \in \Phi$. We denote by $\sigma: \mathbf{W} \rightarrow \mathbf{W}$ the automorphism induced by F . For each $w \in \mathbf{W}$, let $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ be a representative; if $\sigma(w) = w$, then we tacitly assume that $F(\dot{w}) = \dot{w}$.

It is well known that the \mathbf{G}^F -conjugacy classes of F -stable maximal tori of \mathbf{G} are parametrised by the σ -conjugacy classes of \mathbf{W} . Given $w \in \mathbf{W}$, let $g \in \mathbf{G}$ be such that $\dot{w} = g^{-1}F(g)$. (The existence of g relies on Lang’s Theorem, which will be used many times in what follows, without further explicit reference.) Then $\mathbf{T} := g\mathbf{T}_0g^{-1}$ is a corresponding F -stable maximal torus, unique up to conjugation by elements of \mathbf{G}^F ; in this situation, we also say that \mathbf{T} is “of type w ”. (See [4, §2.3] for further details.) For the evaluation of Deligne–Lusztig characters, we shall need to relate \mathbf{G}^F -conjugacy classes of F -stable maximal tori of \mathbf{G} to those in certain connected reductive subgroups of maximal rank. Since this is crucial for the explicit computations that we need to carry out, we will explain the details here; see also Lübeck [30], [31].

2.1. Subsystem subgroups. As in Carter [3, §2], we consider subsets $\Phi' \subseteq \Phi$ that are themselves root systems and are closed in the sense that, whenever $\alpha, \beta \in \Phi'$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi'$. Given such a Φ' , there is a corresponding closed connected reductive subgroup $\mathbf{H}' \subseteq \mathbf{G}$ generated by \mathbf{T}_0

and the subgroups U_α for $\alpha \in \Phi'$. Here, Φ' is the root system of \mathbf{H}' with respect to \mathbf{T}_0 and

$$\mathbf{W}(\Phi') := \langle w_\alpha \mid \alpha \in \Phi' \rangle = N_{\mathbf{H}'}(\mathbf{T}_0)/\mathbf{T}_0$$

is the Weyl group of \mathbf{H}' . Let Ξ be the set of all pairs (Φ', w) where $\Phi' \subseteq \Phi$ is a subset as above and $w \in \mathbf{W}$ is such that $w(\alpha^\dagger) \in \Phi'$ for all $\alpha \in \Phi'$. Given $(\Phi', w) \in \Xi$, we form the corresponding subgroup $\mathbf{H}' \subseteq \mathbf{G}$ as above; then $F(\mathbf{H}') = \dot{w}^{-1}\mathbf{H}'\dot{w}$; note that $\dot{w}U_\alpha\dot{w}^{-1} = U_{w(\alpha)}$ for all $\alpha \in \Phi$. Hence, writing $\dot{w} = g^{-1}F(g)$ for some $g \in \mathbf{G}$, the subgroup $g\mathbf{H}'g^{-1}$ is F -stable and uniquely determined by (Φ', w) , up to conjugation by an element of \mathbf{G}^F . It is known (see Carter [3], Deriziotis [8], Mizuno [48]) that the \mathbf{G}^F -conjugacy classes of F -stable subgroups $g\mathbf{H}'g^{-1}$ as above are parametrised by the pairs in Ξ modulo the equivalence relation defined by: $(\Phi'_1, w_1) \sim (\Phi'_2, w_2)$ if there exists some $x \in \mathbf{W}$ such that $x(\Phi'_1) = \Phi'_2$ and $x^{-1}w_2\sigma(x)w_1^{-1} \in \mathbf{W}(\Phi'_1)$. (The above statement concerning \mathbf{G}^F -conjugacy classes of F -stable maximal tori is a very special case of this correspondence.) Note that $(\Phi', w) \sim (\Phi', uw)$ for all $(\Phi', w) \in \Xi$ and $u \in \mathbf{W}(\Phi')$; thus, one could also consider pairs $(\Phi', \mathbf{W}(\Phi')w)$ but we find it more convenient in our setting here to work with the above definition of Ξ .

2.2. The relation \sim on Ξ . For future reference, we briefly indicate how the relation \sim comes about. (Note that the discussion in [3, §2] assumes that the subgroup \mathbf{H}' corresponding to Φ' is itself F -stable, which will not always be the case if $\sigma \neq \text{id}_{\mathbf{W}}$; furthermore, [3, §2] only considers \mathbf{G}^F -conjugacy for the subgroups corresponding to a fixed Φ' .) So let (Φ'_1, w_1) and (Φ'_2, w_2) be pairs in Ξ ; let $g_1, g_2 \in \mathbf{G}$ be such that $g_1^{-1}F(g_1) = \dot{w}_1$ and $g_2^{-1}F(g_2) = \dot{w}_2$. We have the corresponding F -stable subgroups $g_i\mathbf{H}'_ig_i^{-1}$, where $\mathbf{H}'_i := \langle \mathbf{T}_0, U_\alpha \ (\alpha \in \Phi'_i) \rangle$ for $i = 1, 2$. Suppose now that $g_1\mathbf{H}'_1g_1^{-1}$ and $g_2\mathbf{H}'_2g_2^{-1}$ are conjugate in \mathbf{G}^F ; so $\tilde{g}g_1\mathbf{H}'_1g_1^{-1}\tilde{g}^{-1} = g_2\mathbf{H}'_2g_2^{-1}$ for some $\tilde{g} \in \mathbf{G}^F$. Setting $\hat{g} := g_2^{-1}\tilde{g}g_1$, we have $\hat{g}\mathbf{H}'_1\hat{g}^{-1} = \mathbf{H}'_2$ and there exists some $h_2 \in \mathbf{H}'_2$ such that $\hat{g}\mathbf{T}_0\hat{g}^{-1} = h_2\mathbf{T}_0h_2^{-1}$. Then $n := h_2^{-1}\hat{g} \in N_{\mathbf{G}}(\mathbf{T}_0)$ and $n\mathbf{H}'_1n^{-1} = \mathbf{H}'_2$. Hence, $x(\Phi'_1) = \Phi'_2$, where x is the image of n in \mathbf{W} . We now have $g_1 = \tilde{g}^{-1}g_2\hat{g} = \tilde{g}^{-1}g_2h_2n$ and a straightforward computation yields that

$$\dot{w}_1 = g_1^{-1}F(g_1) = (n^{-1}\dot{w}_2F(n))(F(n)^{-1}\dot{w}_2^{-1}h_2^{-1}\dot{w}_2F(h_2)F(n)).$$

We have $F(\mathbf{H}'_2) = \dot{w}_2^{-1}\mathbf{H}'_2\dot{w}_2$ and so $\dot{w}_2^{-1}h_2^{-1}\dot{w}_2F(h_2) \in F(\mathbf{H}'_2)$. Furthermore, $F(n)^{-1}F(\mathbf{H}'_2)F(n) = F(n^{-1}\mathbf{H}'_2n) = F(\mathbf{H}'_1) = \dot{w}_1^{-1}\mathbf{H}'_1\dot{w}_1$. Hence, we obtain $\dot{w}_1 = n^{-1}\dot{w}_2F(n)\dot{w}_1^{-1}h_1\dot{w}_1$ for some $h_1 \in \mathbf{H}'_1$. Thus, $n^{-1}\dot{w}_2F(n)\dot{w}_1^{-1} \in N_G(\mathbf{T}_0) \cap \mathbf{H}'_1$ and so $x^{-1}w_2\sigma(x)w_1^{-1} \in \mathbf{W}(\Phi'_1)$, as desired. Conversely, if $(\Phi'_1, w_1) \sim (\Phi'_2, w_2)$, then one needs to run the above argument backwards.

2.3. Let us fix a pair $(\Phi', w) \in \Xi$. As already noted, we have $(\Phi', w) \sim (\Phi', uw)$ for all $u \in \mathbf{W}(\Phi')$. By [33, Lemma 1.9], the coset $\mathbf{W}(\Phi')w$ contains a unique element of minimal length; let us denote this element by d . Thus, when considering equivalence classes of pairs $(\Phi', d) \in \Xi$, we may assume without loss of generality that d has minimal length in the coset $\mathbf{W}(\Phi')d$. (Note that Lübeck [30] does not make this assumption on d .) We define a new Frobenius map $F': \mathbf{G} \rightarrow \mathbf{G}$ by $F'(g) := \dot{d}F(g)\dot{d}^{-1}$ for $g \in \mathbf{G}$. Then we have $F'(\mathbf{H}') = \mathbf{H}'$, where $\mathbf{H}' = \langle \mathbf{T}_0, \mathbf{U}_\alpha \ (\alpha \in \Phi') \rangle$. The map induced by F' on \mathbf{W} is given by

$$(a) \quad \sigma': \mathbf{W} \rightarrow \mathbf{W}, \quad w \mapsto d\sigma(w)d^{-1}.$$

Clearly, \mathbf{T}_0 is also an F' -stable maximal torus of \mathbf{H}' . We claim that

$$(b) \quad \mathbf{T}_0 \text{ is a maximally split torus of } \mathbf{H}' \text{ with respect to } F'.$$

This is seen as follows. The group $\mathbf{B}' = \langle \mathbf{T}_0, \mathbf{U}_\alpha \ (\alpha \in \Phi^+ \cap \Phi') \rangle$ is a Borel subgroup of \mathbf{H}' (see [4, §3.5]). Since $\mathbf{T}_0 \subseteq \mathbf{B}'$, it is sufficient to show that \mathbf{B}' is F' -stable. For this purpose, let $\alpha \in \Phi^+ \cap \Phi'$. By [33, Lemma 1.9], we have $d^{-1}(\alpha) \in \Phi^+$ and so $\dot{d}^{-1}\mathbf{U}_\alpha\dot{d} = \mathbf{U}_{d^{-1}(\alpha)} \subseteq \mathbf{B} = F(\mathbf{B})$. Consequently, we have $\mathbf{U}_\alpha \subseteq \dot{d}F(\mathbf{B})\dot{d}^{-1} = F'(\mathbf{B})$. Since also $\mathbf{T}_0 = F'(\mathbf{T}_0) \subseteq F'(\mathbf{B})$, we conclude that $\mathbf{B}' \subseteq F'(\mathbf{B})$. Furthermore, $\mathbf{B}' \subseteq \mathbf{H}' = F'(\mathbf{H}')$ and so $\mathbf{B}' \subseteq F'(\mathbf{B}) \cap F'(\mathbf{H}') = F'(\mathbf{B} \cap \mathbf{H}') = F'(\mathbf{B}')$. Hence, we must have $\mathbf{B}' = F'(\mathbf{B}')$, as claimed. Thus, if Δ' is the unique set of simple roots in $\Phi^+ \cap \Phi'$, then we have $\mathbf{W}(\Phi') = \langle S' \rangle$ where

$$(c) \quad S' := \{w_\alpha \mid \alpha \in \Delta'\} \quad \text{and} \quad \sigma'(S') = S'.$$

In particular, $(\mathbf{W}(\Phi'), S')$ is a Coxeter system and $\sigma'(\mathbf{W}(\Phi')) = \mathbf{W}(\Phi')$.

2.4. In the setting of §2.3, where $(\Phi, d) \in \Xi$, let us also fix an element $g \in \mathbf{G}$ such that $g^{-1}F(g) = d$. Then $\mathbf{T}_d := g\mathbf{T}_0g^{-1} \subseteq \mathbf{G}$ is an F -stable maximal torus of type d . Furthermore, if $\mathbf{H}' = \langle \mathbf{T}_0, \mathbf{U}_\alpha \ (\alpha \in \Phi') \rangle$ and $\mathbf{H}_d := g\mathbf{H}'g^{-1}$, then $\mathbf{T}_d \subseteq \mathbf{H}_d$ and $F(\mathbf{H}_d) = \mathbf{H}_d$. Now Remark 2.3(b) immediately implies that $\mathbf{B}_d := g\mathbf{B}'g^{-1}$ is an F -stable Borel subgroup of \mathbf{H}_d and so \mathbf{T}_d is a maximally split torus of \mathbf{H}_d . Let $\mathbf{W}_d := N_{\mathbf{H}_d}(\mathbf{T}_d)/\mathbf{T}_d$ be the Weyl group of \mathbf{H}_d . We denote by $\sigma_d: \mathbf{W}_d \rightarrow \mathbf{W}_d$ the automorphism induced by F . Then the conjugation map $\gamma_g: \mathbf{G} \rightarrow \mathbf{G}$, $x \mapsto g^{-1}xg$, induces an embedding $\bar{\gamma}_g: \mathbf{W}_d \hookrightarrow \mathbf{W}$, where

$$\mathbf{W}(\Phi') = \bar{\gamma}_g(\mathbf{W}_d) \subseteq \mathbf{W} \quad \text{and} \quad \bar{\gamma}_g \circ \sigma_d = \sigma' \circ \bar{\gamma}_g.$$

Via the isomorphism $\bar{\gamma}_g: \mathbf{W}_d \rightarrow \mathbf{W}(\Phi')$, the σ_d -conjugacy classes of \mathbf{W}_d correspond to the σ' -conjugacy classes of $\mathbf{W}(\Phi')$. Thus, the \mathbf{H}_d^F -conjugacy classes of F -stable maximal tori of \mathbf{H}_d are parametrised by the σ' -conjugacy classes of $\mathbf{W}(\Phi')$. More precisely, if $w' \in \mathbf{W}(\Phi')$, then an F -stable maximal torus $\mathbf{T}' \subseteq \mathbf{H}_d$ of type w' (inside \mathbf{H}_d) is given by $\mathbf{T}' := h\mathbf{T}_d h^{-1}$ where $h \in \mathbf{H}_d$ is such that $h^{-1}F(h) = \gamma_g^{-1}(\dot{w}') = g\dot{w}'g^{-1}$.

The following result describes the fusion from F -stable maximal tori in \mathbf{H}_d to F -stable maximal tori in \mathbf{G} ; see also Lübeck [30, §4.1(2)] (but note that slightly different conventions and assumptions are used in [30]).

Lemma 2.5. *In the above setting, let $\mathbf{T}' \subseteq \mathbf{H}_d$ be an F -stable maximal torus of type $w' \in \mathbf{W}(\Phi')$. Then $\mathbf{T}' \subseteq \mathbf{G}$ is an F -stable maximal torus of type $w'd \in \mathbf{W}$. In particular, a maximally split torus of \mathbf{H}_d is of type d (relative to \mathbf{G}).*

Proof. Recall that $g^{-1}F(g) = \dot{d}$ and that $\mathbf{T}_d = g\mathbf{T}_0g^{-1}$ is a maximally split torus of \mathbf{H}_d . As above, let $h \in \mathbf{H}_d$ be such that $\mathbf{T}' = h\mathbf{T}_d h^{-1}$ and $h^{-1}F(h) = \gamma_g^{-1}(\dot{w}') = g\dot{w}'g^{-1}$. Then $\mathbf{T}' = hg\mathbf{T}_0g^{-1}h^{-1}$ and $(hg)^{-1}F(hg) \in N_{\mathbf{G}}(\mathbf{T}_0)$. Now

$$(hg)^{-1}F(hg) = g^{-1}h^{-1}F(h)F(g) = g^{-1}(g\dot{w}'g^{-1})F(g) = \dot{w}'\dot{d}.$$

Hence, \mathbf{T}' is an F -stable maximal torus of type $\dot{w}'\dot{d}$ in \mathbf{G} . □

Example 2.6. Let \mathbf{G} be simple of type G_2 ; then $\sigma = \text{id}_{\mathbf{W}}$ and the permutation $\alpha \mapsto \alpha^\dagger$ is the identity. Let $\Delta = \{\alpha_1, \alpha_2\}$ be the set of simple roots in Φ^+ , where α_1 is long and α_2 is short. There are two particular subsystems $\Phi' \subseteq \Phi$ that occur in the classification of cuspidal character sheaves on \mathbf{G} (see the proof of [36, Prop. 20.6]). Up to \mathbf{W} -conjugacy, these are Φ'_0 of type $A_1 \times A_1$, spanned by $\{\alpha_2, \alpha_0\}$, and $\Phi''_0 \subseteq \Phi$ of type A_2 , spanned by $\{\alpha_1, \alpha_0\}$. (Here, $\alpha_0 \in \Phi$ denotes the unique root of maximal height in Φ .) There is only one equivalence class of pairs $(\Phi', w) \in \Xi$ under \sim where $\Phi' = \Phi'_0$; a representative is given by (Φ'_0, d_1) with $d_1 = 1_{\mathbf{W}}$. There are two equivalence classes of pairs $(\Phi', w) \in \Xi$ where $\Phi' = \Phi''_0$; representatives are given by (Φ''_0, d_1) with $d_1 = 1_{\mathbf{W}}$, and by (Φ''_0, d_2) with $d_2 = w_{\alpha_2}$ (and d_2 has minimal length in $\mathbf{W}(\Phi''_0)d_2$). The information is summarised in Table 1 (which is a model for the tables in later sections).

In that table, Δ' is the set of simple roots in $\Phi^+ \cap \Phi'$. Furthermore, we define $\sigma'_i \in \text{Aut}(\mathbf{W}(\Phi'))$ by $\sigma'_i(w) = d_i w d_i^{-1}$ for $w \in \mathbf{W}$. Then σ'_i induces a permutation of the simple reflections in $\mathbf{W}(\Phi')$; see Remark 2.3. This permutation, in cycle notation, is indicated in the fourth column of the table; note

Table 1: Subsystems for type G_2

Φ'	Δ'	d_i	permutation	σ'_i -classes
$A_1 \times A_1$	$\alpha_2, 2\alpha_1 + 3\alpha_2$	$d_1 = 1_{\mathbf{W}}$	$()$	4
A_2	$\alpha_1, \alpha_1 + 3\alpha_2$	$d_1 = 1_{\mathbf{W}}$	$()$	3
		$d_2 = w_{\alpha_2}$	$(1, 2)$	3

that this permutation refers to the simple roots in Δ' , not to those in Δ . The last column contains the number of σ'_i -conjugacy classes of $\mathbf{W}(\Phi')$.

Now consider the fusion of F -stable maximal tori described by Lemma 2.5. In each case, we need to work out representatives of the σ'_i -conjugacy classes of $\mathbf{W}(\Phi')$, multiply these by d_i and identify the conjugacy class of \mathbf{W} to which the new element belongs. Here, of course, this can be done by hand, but for larger \mathbf{W} , such computations are conveniently done using the computer algebra system CHEVIE [20], [47], for example.

2.7. Centralisers of semisimple elements. Let C be an F -stable conjugacy class of semisimple elements of \mathbf{G} . It is well-known that $C \cap \mathbf{T}_0$ is non-empty and a single orbit under the action of \mathbf{W} ; furthermore, $C_{\mathbf{G}}^\circ(t)$, for $t \in C \cap \mathbf{T}_0$, is a connected reductive subgroup of the type considered above (see, e.g., [4, §3.5, §3.7]). Thus, we are led to consider the subset $\Xi^\circ \subseteq \Xi$ consisting of all pairs $(\Phi', w) \in \Xi$, for which there exists some $t \in \mathbf{T}_0$ such that

$$\Phi' = \{\alpha \in \Phi \mid \alpha(t) = 1\} \quad \text{and} \quad F(t) = \dot{w}^{-1}tw.$$

Then the \mathbf{G}^F -conjugacy classes of subgroups of the form $C_{\mathbf{G}}^\circ(s)$, where $s \in \mathbf{G}^F$ is semisimple, are parametrised by the pairs in Ξ° modulo the equivalence relation \sim on Ξ° . (See again [3], [8], [48].) Given a pair $(\Phi', w) \in \Xi^\circ$, a corresponding semisimple element $s \in \mathbf{G}^F$ is obtained as follows. Let $t \in \mathbf{T}_0$ be such that $\Phi' = \{\alpha \in \Phi \mid \alpha(t) = 1\}$ and $F(t) = \dot{w}^{-1}tw$; then $C_{\mathbf{G}}^\circ(t) = \langle \mathbf{T}_0, \mathbf{U}_\alpha (\alpha \in \Phi') \rangle$. Let $g \in \mathbf{G}$ be such that $g^{-1}F(g) = \dot{w}$ and set $s := gtg^{-1}$. Then $F(s) = s$ and $C_{\mathbf{G}}^\circ(s) = gC_{\mathbf{G}}^\circ(t)g^{-1}$.

Note that, if $d \in \mathbf{W}$ has minimal length in the coset $\mathbf{W}(\Phi')w$, then $w = w'd$ for some $w' \in \mathbf{W}(\Phi')$ and we still have $F(t) = \dot{d}^{-1}td$ (since $\dot{w}' \in C_{\mathbf{G}}(t)$). Hence, again, we may assume without loss of generality that $w = d$ and so the discussions in Remarks 2.3, 2.4, and Lemma 2.5 apply. The subsystems $\Phi' \subseteq \Phi$ which can arise at all as the root system of $C_{\mathbf{G}}^\circ(t)$, for some $t \in \mathbf{T}_0$, are characterised in [8, §2.3]; given such a subset $\Phi' \subseteq \Phi$, the condition of whether there is some $w \in \mathbf{W}$ such that $(\Phi', w) \in \Xi^\circ$ may also depend on the isogeny type of \mathbf{G} and the \mathbb{F}_q -rational structure on \mathbf{G} ; see [3, §5] and [8, Chap. 2] for further details.

3. On the evaluation of Deligne–Lusztig characters

Let $\mathbf{T} \subseteq \mathbf{G}$ be an F -stable maximal torus and $\theta \in \text{Irr}(\mathbf{T}^F)$ be an irreducible character. Then we have a corresponding virtual character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ of \mathbf{G}^F , as defined by Deligne–Lusztig [7] (see also [4, Chap. 7]). These virtual characters span a significant subspace of the space of all class functions on \mathbf{G}^F (see the introduction of [33] and [21, Cor. 2.7.13] for a more precise measure of what “significant” means). It is known that, if $u \in \mathbf{G}^F$ is unipotent, then $Q_{\mathbf{T}}^{\mathbf{G}}(u) := R_{\mathbf{T}}^{\mathbf{G}}(\theta)(u) \in \mathbb{Z}$ does not depend on θ ; the function $u \mapsto Q_{\mathbf{T}}^{\mathbf{G}}(u)$ is called a *Green function*. We now have the following important character formula.

Let $\tilde{g} \in \mathbf{G}^F$ and write $\tilde{g} = su = us$, where $s \in \mathbf{G}^F$ is semisimple and $u \in \mathbf{G}^F$ is unipotent. Then, setting $\mathbf{H}_s := C_{\mathbf{G}}^{\circ}(s)$, we have

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(\tilde{g}) = \frac{1}{|\mathbf{H}_s^F|} \sum_{x \in \mathbf{G}^F : x^{-1}sx \in \mathbf{T}} Q_{x\mathbf{T}x^{-1}}^{\mathbf{H}_s}(u) \theta(x^{-1}sx).$$

Note that, firstly, \mathbf{H}_s is connected and reductive; secondly, if $x^{-1}sx \in \mathbf{T}^F$, then $x\mathbf{T}x^{-1}$ is an F -stable maximal torus contained in \mathbf{H}_s ; furthermore, u is known to belong to \mathbf{H}_s . (See [7, 4.2] or [4, 7.2.8] for further details.) In particular, the formula shows that all values of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ belong to the field $\mathbb{Q}(\theta(t) \mid t \in \mathbf{T}^F)$. We also see that, if s is not conjugate in \mathbf{G}^F to an element of \mathbf{T}^F , then $R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = 0$.

We now explain how the above formula can be evaluated explicitly; for this purpose, we need to

- (1) know the values of the Green functions of \mathbf{H}_s ,
- (2) deal with the sum over all $x \in \mathbf{G}^F$ such that $x^{-1}sx \in \mathbf{T}$.

As far as (1) is concerned, see the surveys in [10, Chap. 13] and [21, §2.8]. In any case, for \mathbf{G} simple of exceptional type, explicit tables are known by Beynon–Spaltenstein [1] and Shoji [51]. (The fact that these tables remain valid whenever p is a “good” prime for \mathbf{G} follows from [54, Theorem 5.5]; see also [19] for “bad” p .) The tables can be obtained via the function `ICCTable` of Michel’s version of `CHEVIE` [47]. As far as (2) is concerned, the following result provides a first simplification.

Lemma 3.1 (See [21, 2.2.23]). *In the above setting, assume that s is conjugate in \mathbf{G}^F to an element in \mathbf{T}^F . Let $\mathbf{T}_1, \dots, \mathbf{T}_m$ be representatives of the \mathbf{H}_s^F -conjugacy classes of F -stable maximal tori of \mathbf{H}_s that are conjugate in \mathbf{G}^F to \mathbf{T} . For each i , let $\tilde{g}_i \in \mathbf{G}^F$ be such that $\mathbf{T}_i = \tilde{g}_i\mathbf{T}\tilde{g}_i^{-1}$. Then*

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(\tilde{g}) = \sum_{1 \leq i \leq m} Q_{\mathbf{T}_i}^{\mathbf{H}_s}(u) \frac{1}{|\mathbf{W}(\mathbf{H}_s, \mathbf{T}_i)^F|} \sum_{y \in \mathbf{W}(\mathbf{G}, \mathbf{T}_i)^F} \theta(\tilde{g}_i^{-1}y^{-1}sy\tilde{g}_i)$$

where $\mathbf{W}(\mathbf{G}, \mathbf{T}_i) = N_{\mathbf{G}}(\mathbf{T}_i)/\mathbf{T}_i$ and $\mathbf{W}(\mathbf{H}_s, \mathbf{T}_i) = N_{\mathbf{H}_s}(\mathbf{T}_i)/\mathbf{T}_i$.

Now note that the subgroup $\mathbf{T}^F \subseteq \mathbf{G}^F$ is not really computationally accessible, and the same is true for $\mathbf{T}_1^F, \dots, \mathbf{T}_m^F$. All we can do explicitly are computations within \mathbf{T}_0 . Hence, in order to proceed, we use a different model for $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, as already constructed in [7]. Assume that \mathbf{T} is of type $w \in \mathbf{W}$ and let $g_1 \in \mathbf{G}$ be such that $g_1^{-1}F(g_1) = \dot{w}$. Then define

$$\mathbf{T}_0[w] := \{t \in \mathbf{T} \mid F(t) = \dot{w}^{-1}t\dot{w}\} = g_1^{-1}\mathbf{T}^F g_1 \subseteq \mathbf{G}.$$

Let $\theta' \in \text{Irr}(\mathbf{T}_0[w])$ be the irreducible character defined by $\theta'(t) := \theta(g_1 t g_1^{-1})$ for all $t \in \mathbf{T}_0[w]$. Then we have $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = R_w^{\theta'}$, where the right hand side is constructed directly from (w, θ') (see [21, 2.3.18] for further details). Thus, the virtual characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ may equally well be defined in terms of pairs (w, θ') , where $w \in \mathbf{W}$ and $\theta' \in \text{Irr}(\mathbf{T}_0[w])$ — and the latter set of pairs (w, θ') is, indeed, computationally accessible.

We can now apply the results in the previous section, especially the discussions in Remark 2.4 and Lemma 2.5. Let $\tilde{g} = su = us \in \mathbf{G}^F$ as above and $\mathbf{H}_s := C_{\mathbf{G}}^{\circ}(s)$. Let $\mathbf{T}' \subseteq \mathbf{H}_s$ be a maximally split torus and $g \in \mathbf{G}$ be such that $\mathbf{T}' = g\mathbf{T}_0 g^{-1}$. Then $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T}_0)$ and we denote by $d \in \mathbf{W}$ the image of $g^{-1}F(g)$ in \mathbf{W} . Let $t := g^{-1}sg \in \mathbf{T}_0$ and $\Phi' := \{\alpha \in \Phi \mid \alpha(t) = 1\}$. Then $F(t) = \dot{w}^{-1}t\dot{w}$ and $(\Phi', d) \in \Xi^{\circ}$ parametrises the \mathbf{G}^F -conjugacy class of \mathbf{H}_s ; furthermore, we have that d has minimal length in $\mathbf{W}(\Phi')d$. As in Remark 2.3, we define $\sigma' \in \text{Aut}(\mathbf{W})$ by $\sigma'(w) = d\sigma(w)d^{-1}$ for $w \in \mathbf{W}$; then $\sigma'(\mathbf{W}(\Phi')) = \mathbf{W}(\Phi')$.

- (*) Let $w'_1, \dots, w'_m \in \mathbf{W}(\Phi')$ be representatives of the σ' -conjugacy classes of $\mathbf{W}(\Phi')$ such that $w'_i d$ is σ -conjugate in \mathbf{W} to w . For each i let us fix an element $x_i \in \mathbf{W}$ such that $w = x_i^{-1}w'_i d \sigma(x_i)$.

For $1 \leq i \leq m$ let $h_i \in \mathbf{H}_s$ be such that $h_i^{-1}F(h_i) = gw'_i g^{-1}$, and set $\mathbf{T}_i := h_i \mathbf{T}' h_i^{-1} \subseteq \mathbf{H}_d$. Then, by Lemma 2.5, $\mathbf{T}_1, \dots, \mathbf{T}_m$ are maximal tori as required in Lemma 3.1. For $1 \leq i \leq m$ define $\mathbf{T}_0[w'_i d] \subseteq \mathbf{T}_0$ analogously to $\mathbf{T}_0[w]$ above; then $\mathbf{T}_0[w'_i d] = \dot{x}_i \mathbf{T}_0[w] \dot{x}_i^{-1}$ (see [21, 2.3.20]). So, given $\theta' \in \text{Irr}(\mathbf{T}_0[w])$ as above, we can define a character $\theta'_i \in \text{Irr}(\mathbf{T}_0[w'_i d])$ by

$$\theta'_i(t) := \theta'(\dot{x}_i^{-1}t\dot{x}_i) \quad \text{for } t \in \mathbf{T}_0[w'_i d].$$

Now let $C_{\mathbf{W},\sigma}(w) = \{x \in \mathbf{W} \mid xw = w\sigma(x)\}$ be the σ -centraliser of w in \mathbf{W} ; then $\dot{x}\mathbf{T}_0[w]\dot{x}^{-1} = \mathbf{T}_0[w]$ for all $x \in C_{\mathbf{W},\sigma}(w)$. Defining $C_{\mathbf{W},\sigma}(w'_i d)$ analogously, we have $\dot{x}\mathbf{T}_0[w'_i d]\dot{x}^{-1} = \mathbf{T}_0[w'_i d]$ for all $x \in C_{\mathbf{W},\sigma}(w'_i d)$. Let also

$$C_{\mathbf{W}(\Phi'),\sigma'}(w'_i) = \{x \in \mathbf{W}(\Phi') \mid xw'_i = w'_i\sigma'(x)\} = \mathbf{W}(\Phi') \cap C_{\mathbf{W},\sigma}(w'_i d).$$

With this notation, we can now state the following result; see also Lübeck [30, Satz 2.1] for a slightly different formulation.

Lemma 3.2. *In the above setting, we have $\mathbf{W}(\mathbf{H}_s, \mathbf{T}_i)^F \cong C_{\mathbf{W}(\Phi'), \sigma'}(w'_i)$ and*

$$\sum_{y \in \mathbf{W}(\mathbf{G}, \mathbf{T}_i)^F} \theta(\tilde{g}_i^{-1} y^{-1} s y \tilde{g}_i) = \sum_c \theta'_i(\dot{c}^{-1} t \dot{c}) \quad \text{for } 1 \leq i \leq m,$$

where c runs over all elements of $C_{\mathbf{W}, \sigma}(w'_i d)$. In particular, if $\theta = 1_{\mathbf{T}}$ is the trivial character of \mathbf{T}^F , then the above sum equals $|C_{\mathbf{W}, \sigma}(w'_i d)|$ for $1 \leq i \leq m$.

Proof. Recall that $\mathbf{T} = g_1 \mathbf{T}_0 g_1^{-1}$, $\mathbf{T}' = g \mathbf{T}_0 g^{-1}$ and $\mathbf{T}_i = h_i \mathbf{T}' h_i^{-1}$ for all i . Hence, setting $\tilde{g}_i := h_i g \dot{x}_i g_1^{-1} \in \mathbf{G}$, we have $\tilde{g}_i \mathbf{T} \tilde{g}_i^{-1} = \mathbf{T}_i$ for all i . Since \mathbf{T} and \mathbf{T}_i are \mathbf{G}^F -conjugate, we can replace h_i by $h_i t'_i$ for a suitable $t'_i \in \mathbf{T}'$ such that $F(\tilde{g}_i) = \tilde{g}_i$ (see the argument in the proof of [4, Prop. 3.3.3]). Thus, the elements \tilde{g}_i are as required in Lemma 3.1. Next, by [4, Prop. 3.3.6], we have a group isomorphism

$$C_{\mathbf{W}, \sigma}(w'_i d) \xrightarrow{\sim} \mathbf{W}(\mathbf{G}, \mathbf{T}_i)^F, \quad c \mapsto h_i g \dot{c} g^{-1} h_i^{-1}.$$

(Recall that $h_i g \mathbf{T}_0 g^{-1} h_i^{-1} = \mathbf{T}_i$ and $(h_i g)^{-1} F(h_i g) = \dot{w}'_i \dot{d}$.) Hence, we have

$$\sum_{y \in \mathbf{W}(\mathbf{G}, \mathbf{T}_i)^F} \theta(\tilde{g}_i^{-1} y^{-1} s y \tilde{g}_i) = \sum_c \theta(\tilde{g}_i^{-1} h_i g \dot{c}^{-1} g^{-1} h_i^{-1} s h_i g \dot{c} g^{-1} h_i^{-1} \tilde{g}_i)$$

where c runs over all elements of $C_{\mathbf{W}, \sigma}(w'_i d)$. Now $g^{-1} h_i^{-1} s h_i g = t$; hence, the terms in the above sum on the right hand side are given by

$$\theta(\tilde{g}_i^{-1} h_i g \dot{c}^{-1} t \dot{c} g^{-1} h_i^{-1} \tilde{g}_i) = \theta(g_1 \dot{x}_i^{-1} \dot{c}^{-1} t \dot{c} \dot{x}_i g_1^{-1}) = \theta'_i(\dot{c}^{-1} t \dot{c})$$

for all $c \in C_{\mathbf{W}, \sigma}(w'_i d)$, as required. Finally, consider the assertion concerning $\mathbf{W}(\mathbf{H}_s, \mathbf{T}_i)^F$. Let $\mathbf{W}_s = N_{\mathbf{H}_s}(\mathbf{T}')/\mathbf{T}'$ and $\sigma_s: \mathbf{W}_s \rightarrow \mathbf{W}_s$ be induced by F . Let $\mathbf{H}' = C_{\mathbf{G}}^\circ(t) = g^{-1} \mathbf{H}_s g$ and $F': \mathbf{H}' \rightarrow \mathbf{H}'$ be as in Remark 2.3; recall that F' induces $\sigma' \in \text{Aut}(\mathbf{W}(\Phi'))$. As discussed in Remark 2.4, conjugation by g induces a bijection between the σ' -conjugacy classes in $\mathbf{W}(\Phi')$ and the σ_s -conjugacy classes in \mathbf{W}_s . Thus, we have $\mathbf{W}(\mathbf{H}_s, \mathbf{T}_i)^F \cong \mathbf{W}(\mathbf{H}', \mathbf{T}'_i)^{F'}$ where $\mathbf{T}'_i \subseteq \mathbf{H}'$ is an F' -stable maximal torus of type $w'_i \in \mathbf{W}(\Phi')$ (relative to F'). Again by [4, Prop. 3.3.6], the group $\mathbf{W}(\mathbf{H}', \mathbf{T}'_i)^{F'}$ is isomorphic to $C_{\mathbf{W}(\Phi'), \sigma'}(w'_i)$. \square

The point about the above result is that the formula on the right hand side of the identity can be explicitly and effectively computed, once a character $\theta' \in \text{Irr}(\mathbf{T}_0[w])$ has been specified: all we need to know is the action of \mathbf{W} on \mathbf{T}_0 , plus information concerning various σ -conjugacy classes in \mathbf{W} .

Example 3.3. Assume that $\tilde{g} = su = us$ where $u \in \mathbf{H}_s$ is regular unipotent. Then $Q_{\mathbf{T}_i}^{\mathbf{H}_s}(u) = 1$ for $1 \leq i \leq m$ (see [7, Theorem 9.16]). Let $\theta = 1_{\mathbf{T}}$ be the trivial character of \mathbf{T}^F . Then

$$R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}})(g) = |C_{\mathbf{W},\sigma}(w)| \sum_{1 \leq i \leq m} |C_{\mathbf{W}(\Phi'),\sigma'}(w'_i)|^{-1}.$$

Indeed, this is now clear by Lemmas 3.1 and 3.2. Note that, since the maximal tori \mathbf{T} and \mathbf{T}_i are conjugate in \mathbf{G}^F , we have $\mathbf{W}(\mathbf{G}, \mathbf{T})^F \cong \mathbf{W}(\mathbf{G}, \mathbf{T}_i)^F$ and, hence, $\mathbf{W}(\mathbf{G}, \mathbf{T}_i)^F \cong C_{\mathbf{W},\sigma}(w)$ for $1 \leq i \leq m$.

Example 3.4. Assume that \mathbf{G} is of split type; then $\sigma = \text{id}_{\mathbf{W}}$. Then we define

$$R_{\phi}^{\mathbf{G}} := \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \phi(w) R_{\mathbf{T}_w}^{\mathbf{G}}(1) \quad \text{for } \phi \in \text{Irr}(\mathbf{W}),$$

where $\mathbf{T}_w \subseteq \mathbf{G}$ is an F -stable maximal torus of type w and 1 stands for the trivial character of \mathbf{T}_w^F . (This is a very special case of [33, (3.7.1)].) We also have

$$R_{\mathbf{T}_w}^{\mathbf{G}}(1) = \sum_{\phi \in \text{Irr}(\mathbf{W})} \phi(w) R_{\phi}^{\mathbf{G}} \quad \text{for all } w \in \mathbf{W};$$

so knowing the $R_{\mathbf{T}_w}^{\mathbf{G}}(1)$'s is equivalent to knowing the $R_{\phi}^{\mathbf{G}}$'s. Also assume now that $\tilde{g} = su = us$ is such that $s \in \mathbf{T}_0^F$. Then one easily sees that

$$R_{\phi}^{\mathbf{G}}(\tilde{g}) = \sum_{\psi \in \text{Irr}(\mathbf{W}_s)} m(\psi, \phi) R_{\psi}^{\mathbf{H}_s}(u) \quad \text{for any } \phi \in \text{Irr}(\mathbf{W}),$$

where $\mathbf{W}_s = N_{\mathbf{H}_s}(\mathbf{T}_0)/\mathbf{T}_0 \subseteq \mathbf{W}$ is the Weyl group of \mathbf{H}_s and $m(\psi, \phi)$ denotes the multiplicity of $\psi \in \text{Irr}(\mathbf{W}_s)$ in the restriction of ϕ . Thus, here the question of finding the fusion of F -stable maximal tori from \mathbf{H}_s to \mathbf{G} has been absorbed into the question of decomposing the restriction of any $\phi \in \text{Irr}(\mathbf{W})$ to \mathbf{W}_s .

Example 3.5. Let \mathbf{G} be simple of type F_4 , where $p \neq 2$. There exists an involution $s \in \mathbf{T}_0^F$ such that $\mathbf{H}' := C_{\mathbf{G}}(s)$ has a root system of type B_4 (see §7.3 below for further details). Furthermore, there is a unipotent element $u \in \mathbf{H}^F$ such that, if we let Σ be the \mathbf{G}^F -conjugacy class of su , then Σ^F splits into five conjugacy classes in \mathbf{G}^F , with centraliser orders $8q^8, 8q^8, 4q^4, 4q^4, 4q^4$ (see §7.10 for more details). The condition on the centraliser orders uniquely determines the conjugacy class of u in \mathbf{H}' . Now let $u' \in \mathbf{H}'^F$ be one of the unipotent elements such that $su' \in \Sigma$ and $|C_{\mathbf{G}}(su')^F| = 8q^8$.

Let $\mathbf{W}' \subseteq \mathbf{W}$ be the Weyl group of \mathbf{H}' . Then $\text{Irr}(\mathbf{W}')$ is parametrised by the bi-partitions of 4. By the output of the function `ICCTable` in Michel's version of CHEVIE [47], the only $\psi \in \text{Irr}(\mathbf{W}')$ such that $R_{\psi}^{\mathbf{H}'}(u') \neq 0$ are $\psi_{(4,-)}$, $\psi_{(3,1)}$, $\psi_{(-,4)}$, $\psi_{(22,-)}$, $\psi_{(2,2)}$. Furthermore, we have

$$R_{\psi_{(4,-)}}^{\mathbf{H}'}(u') = 1, R_{\psi_{(3,1)}}^{\mathbf{H}'}(u') = q, R_{\psi_{(-,4)}}^{\mathbf{H}'}(u') = R_{\psi_{(22,-)}}^{\mathbf{H}'}(u') = R_{\psi_{(2,2)}}^{\mathbf{H}'}(u') = q^2.$$

In order to evaluate the formula for $R_{\phi}^{\mathbf{G}}$ in Example 3.4, we need to know the multiplicities $m(\psi, \phi)$ for $\psi \in \text{Irr}(\mathbf{W}')$ and $\phi \in \text{Irr}(\mathbf{W})$; these are readily available via the function `InductionTable` of CHEVIE [47]. This yields the following values:

$$\begin{aligned} R_{\phi_{9,6}''}^{\mathbf{G}}(su') &= R_{\phi_{4,8}}^{\mathbf{G}}(su') = R_{\phi_{1,12}''}^{\mathbf{G}}(su') = q^2, \\ R_{\phi_{12,4}}^{\mathbf{G}}(su') &= R_{\phi_{9,6}'}^{\mathbf{G}}(su') = R_{\phi_{1,12}}^{\mathbf{G}}(su') = 0, \end{aligned}$$

where we use the notation of Carter [4, p. 413] for the irreducible characters of \mathbf{W} . (CHEVIE uses the same notation; the conversion to the notation defined and used by Lusztig [33] is displayed in Table 2.) The knowledge of the above values will turn out to be useful in the further discussion in §7.10.

Table 2: Conventions for the labelling of $\text{Irr}(\mathbf{W})$ for type F_4

$\phi_{1,0}$	$\phi_{1,12}''$	$\phi'_{1,12}$	$\phi_{1,24}$	$\phi_{2,4}''$	$\phi'_{2,16}$	$\phi'_{2,4}$	$\phi_{2,16}''$	$\phi_{4,8}$	$\phi_{9,2}$	$\phi_{9,6}''$	$\phi'_{9,6}$	$\phi_{9,10}$
1_1	1_3	1_2	1_4	2_3	2_4	2_1	2_2	4_1	9_1	9_3	9_2	9_4
$\phi'_{6,6}$	$\phi_{6,6}''$	$\phi_{12,4}$	$\phi_{4,1}$	$\phi_{4,7}''$	$\phi'_{4,7}$	$\phi_{4,13}$	$\phi_{8,3}''$	$\phi'_{8,9}$	$\phi'_{8,3}$	$\phi_{8,9}''$	$\phi_{16,5}$	
6_1	6_2	12_1	4_2	4_4	4_3	4_5	8_3	8_4	8_1	8_2	16_1	

The labels $\phi_{1,0}$ etc. are those in [4, p. 413];
 the labels 1_1 etc. those of Lusztig [33, 4.10].

4. Characteristic functions and conjugacy classes

Let $\hat{\mathbf{G}}$ denote the set of character sheaves on \mathbf{G} (up to isomorphism), as defined by Lusztig [35]. If $A \in \hat{\mathbf{G}}$ is F -invariant, that is, we have $F^*A \cong A$, then the choice of an isomorphism $\phi: F^*A \xrightarrow{\sim} A$ gives rise to a characteristic function $\chi_A: \mathbf{G}^F \rightarrow \overline{\mathbb{Q}}_l$ (where $l \neq p$ is a prime); see [39, §5]. The isomorphism ϕ can be chosen such that the values of χ_A are cyclotomic integers and the standard inner product of χ_A with itself is 1. Hence, we may assume that χ_A is a function with values in \mathbb{K} . The various functions arising in this way form

an orthonormal basis of the space of class functions on \mathbf{G}^F . (See [37, §25]; these results hold unconditionally because of the “cleanness” established in [44].) Similarly to the situation for $\text{Irr}(\mathbf{G}^F)$, we have a partition $\hat{\mathbf{G}} = \coprod_s \hat{\mathbf{G}}_s$ where s runs over the semisimple elements (up to conjugation) in a group \mathbf{G}^* dual to \mathbf{G} (see [41, 1.2].) The character sheaves in $\hat{\mathbf{G}}_s$, where $s = 1$, are called unipotent character sheaves.

In the case where A is a cuspidal character sheaf (and \mathbf{G} is simple), the characteristic functions χ_A can be evaluated in a simple way. We begin with some general remarks concerning conjugacy classes.

4.1. Parametrisation of \mathbf{G}^F -conjugacy classes. Let Σ be an F -stable conjugacy class of \mathbf{G} . Let us fix a representative $g_1 \in \Sigma^F$ and set $A_{\mathbf{G}}(g_1) := C_{\mathbf{G}}(g_1)/C_{\mathbf{G}}^\circ(g_1)$. Then F induces an automorphism of $A_{\mathbf{G}}(g_1)$ that we denote by the same symbol. Given $a \in A_{\mathbf{G}}(g_1)$, let $\dot{a} \in C_{\mathbf{G}}(g_1)$ be a representative of a and write $\dot{a} = x^{-1}F(x)$ for some $x \in \mathbf{G}$. Then $g_a := xg_1x^{-1} \in \mathbf{G}^F$; let C_a be the \mathbf{G}^F -conjugacy class of g_a . A standard argument (using Lang’s Theorem, see [58, I, 2.7]) shows that C_a only depends on a ; furthermore $\Sigma^F = \bigcup_{a \in A_{\mathbf{G}}(g_1)} C_a$, where $C_a = C_{a'}$ if and only if a, a' are F -conjugate in $A_{\mathbf{G}}(g_1)$. Now there are two natural operations on the \mathbf{G}^F -conjugacy classes contained in Σ^F .

(a) The first one is denoted by $\text{Sh}_{\mathbf{G}}$ and called the Shintani map. Let C be a \mathbf{G}^F -conjugacy class contained in Σ^F ; thus, $C = C_a$ for some $a \in A_{\mathbf{G}}(g_1)$. Let $g \in C$ and write $g = x^{-1}F(x)$ for some $x \in \mathbf{G}$. Then $g' := xgx^{-1} \in \mathbf{G}^F$ and the \mathbf{G}^F -conjugacy class of g' does not depend on the choice of g or x ; we denote that class by $\text{Sh}_{\mathbf{G}}(C)$. By Digne–Michel [9, Chap. IV, Prop. 1.1], we have

$$\text{Sh}_{\mathbf{G}}(C_a) = C_{\bar{g}_1 a} \quad (a \in A_{\mathbf{G}}(g_1))$$

where \bar{g}_1 denotes the image of $g_1 \in C_{\mathbf{G}}(g_1)$ in $A_{\mathbf{G}}(g_1)$.

(b) The second one, defined in [41, 3.1], only plays a role when $Z(\mathbf{G}) \neq \{1\}$. Let $z \in Z(\mathbf{G})$ and write $z = t^{-1}F(t)$ for some $t \in \mathbf{G}$. (We could even take $t \in \mathbf{T}_0$.) As above, let $C = C_a$ be a \mathbf{G}^F -conjugacy class contained in Σ^F . One easily sees that $\gamma_z(C) := tCt^{-1}$ is a conjugacy class in \mathbf{G}^F and does not depend on the choice of t ; furthermore, we have

$$\gamma_z(C_a) = C_{\bar{z}a} \quad (a \in A_{\mathbf{G}}(g_1))$$

where $\bar{z} \in A_{\mathbf{G}}(g_1)$ denotes the image of z under the natural map $Z(\mathbf{G}) \rightarrow A_{\mathbf{G}}(g_1)$.

4.2. Characteristic functions of cuspidal character sheaves. Assume that \mathbf{G} is a simple algebraic group. Let A be a cuspidal character sheaf on \mathbf{G} such that $F^*A \cong A$. (See [35, Def. 3.10]; such an A may be unipotent or not.) Then there exists an F -stable conjugacy class Σ of \mathbf{G} and an irreducible, \mathbf{G} -equivariant $\overline{\mathbb{Q}}_l$ -local system \mathcal{E} on Σ such that $F^*\mathcal{E} \cong \mathcal{E}$ and $A = \mathrm{IC}(\overline{\Sigma}, \mathcal{E})[\dim \Sigma]$; see [35, 3.12]. Let us fix $g_1 \in \Sigma^F$ and set $A_{\mathbf{G}}(g_1) := C_{\mathbf{G}}(g_1)/C_{\mathbf{G}}^{\circ}(g_1)$, as above. We further assume that:

- (*) the local system \mathcal{E} is one-dimensional and, hence, corresponds to an F -invariant linear character $\psi: A_{\mathbf{G}}(g_1) \rightarrow \mathbb{K}^{\times}$ (via [42, 19.7]).

(This assumption will be satisfied in all examples that we consider.) Now (*) implies that the function $\psi: A_{\mathbf{G}}(g_1) \rightarrow \mathbb{K}^{\times}$ is constant on the F -conjugacy classes of $A_{\mathbf{G}}(g_1)$. Hence, we obtain a class function $\chi_{g_1, \psi}: \mathbf{G}^F \rightarrow \mathbb{K}$ by setting

$$\chi_{g_1, \psi}(g) := \begin{cases} q^{(\dim \mathbf{G} - \dim \Sigma)/2} \psi(a) & \text{if } g \in C_a \text{ for some } a \in A_{\mathbf{G}}(g_1), \\ 0 & \text{if } g \notin \Sigma^F. \end{cases}$$

(Note that there are cases where $\dim \mathbf{G} - \dim \Sigma$ is not even; in such a case, we also need to fix a square root of q in \mathbb{K} . This is typically done by fixing once and for all a square root $\sqrt{p} \in \mathbb{K}$ and then setting $\sqrt{q} := \sqrt{p}^f$ if $q = p^f$ with $f \geq 1$.) Since \mathcal{E} is one-dimensional, we can choose an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ such that the induced map on the stalk \mathcal{E}_{g_1} is given by scalar multiplication by $q^{(\dim \mathbf{G} - \dim \Sigma)/2}$. Then this isomorphism canonically induces an isomorphism $\phi: F^*A \xrightarrow{\sim} A$ and $\chi_{g_1, \psi}$ is the corresponding characteristic function χ_A , of norm 1 with respect to the standard inner product. (This follows from the fact that A is “clean” [44], using the construction in [42, 19.7].) We shall also set

$$\lambda_A := \psi(\bar{g}_1) \quad \text{where } \bar{g}_1 \text{ denotes the image of } g_1 \text{ in } A_{\mathbf{G}}(g_1).$$

Then λ_A is a root of unity that only depends on A (see Shoji [53, Theorem 3.3], [53, Prop. 3.8]); it is a useful invariant of A . In this context, we have the following basic problem, formulated by Lusztig [41, 0.4(a)]:

- (♣) Express the functions $\chi_{g_1, \psi}$ as explicit linear combinations of $\mathrm{Irr}(\mathbf{G}^F)$.

This problem is solved in many cases, but not in complete generality. Some examples in small rank cases (types $A_1, C_2, {}^3D_4, \dots$) are mentioned in [17, Example 7.8]. We will produce further examples below.

For \mathbf{G} simple of exceptional type, many cuspidal character sheaves turn out to be unipotent. (Exceptions only occur in types E_6 and E_7 .) So it is of particular importance to address these cases.

4.3. Cuspidal unipotent character sheaves. Assume that \mathbf{G} is simple, of split type (so $\sigma = \text{id}_{\mathbf{W}}$). Let $\text{Unip}(\mathbf{G}^F)$ denote the set of unipotent characters of \mathbf{G}^F . By [33, Main Theorem 4.23], $\text{Unip}(\mathbf{G}^F)$ is parametrised by a certain set $X(\mathbf{W})$ which only depends on \mathbf{W} ; we shall write $\text{Unip}(\mathbf{G}^F) = \{\rho_x \mid x \in X(\mathbf{W})\}$. For each $x \in X(\mathbf{W})$, we also have a corresponding almost character R_x , defined as an explicit linear combination of $\text{Unip}(\mathbf{G}^F)$; see [33, 4.24.1]. The matrix of multiplicities $\langle \rho_x, R_{x'} \rangle$ ($x, x' \in X(\mathbf{W})$) is Lusztig’s non-abelian Fourier matrix; see also [4, §13.6]. There is an embedding $\text{Irr}(\mathbf{W}) \hookrightarrow X(\mathbf{W})$, $\phi \mapsto x_\phi$, such that

$$R_{x_\phi} = R_\phi^{\mathbf{G}} = \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \phi(w) R_{T_w}^{\mathbf{G}}(1) \quad \text{for } \phi \in \text{Irr}(\mathbf{W}).$$

Thus, the values of R_{x_ϕ} can be computed using the character table of \mathbf{W} and the results discussed in Section 3. Finally, the unipotent character sheaves on \mathbf{G} are also parametrised by $X(\mathbf{W})$; see [37, Theorem 23.1] (plus the “cleanness” in [44]). If $x \in X(\mathbf{W})$ is such that A_x is cuspidal and F -invariant, then we have a corresponding characteristic function $\chi_{g_1, \psi}$ as in §4.2. In this situation, there is the following solution of (♣) in §4.2. For all $x \in X(\mathbf{W})$ such that A_x is cuspidal, we have

$$(\clubsuit) \quad R_x = \zeta \chi_{g_1, \psi} \text{ for some scalar } \zeta \in \mathbb{K} \text{ of absolute value } 1.$$

If p is sufficiently large, then this is part of Lusztig [41, Theorem 0.8]. For arbitrary p , this is part of the main results of Shoji [53], [54], (The latter results hold without condition on p , thanks to the “cleanness” in [44].) The scalars ζ are determined by Shoji [52], [56] for \mathbf{G} of classical type. For exceptional types, there are a number of cases where the scalars ζ remain to be determined, and it is one purpose of this paper to reduce the number of open cases.

The following technical result will be needed in Section 6.

Lemma 4.4. *In the setting of §4.1, let $a \in A_{\mathbf{G}}(g_1)$ and $z \in Z(\mathbf{G})$. Then every $\rho \in \text{Unip}(\mathbf{G}^F)$ takes the same value on C_a and on $C_{\bar{z}a}$.*

Proof. Let $g \in C_a$. By §4.1(b) we have $C_{\bar{z}a} = \gamma_z(C_a)$. Hence, writing $z = t^{-1}F(t)$ for some $t \in \mathbf{G}$, we have $g' := tgt^{-1} \in C_{\bar{z}a}$. Let $\rho \in \text{Unip}(\mathbf{G}^F)$. In order to show that $\rho(g) = \rho(g')$, we use a regular embedding $\mathbf{G} \subseteq \tilde{\mathbf{G}}$ (see, e.g., [21, §1.7]). Thus, $\tilde{\mathbf{G}}$ is a connected reductive group with a connected center and $\mathbf{G}, \tilde{\mathbf{G}}$ have the same derived subgroup; furthermore, $\tilde{\mathbf{G}}$ is also defined over \mathbb{F}_q and we denote the corresponding Frobenius map again by F . Now $Z(\mathbf{G}) \subseteq Z(\tilde{\mathbf{G}})$ and so, since $Z(\tilde{\mathbf{G}})$ is connected, we can write $z = \tilde{t}^{-1}F(\tilde{t})$ where $\tilde{t} \in Z(\tilde{\mathbf{G}})$. Then $h := \tilde{t}\tilde{t}^{-1} \in \tilde{\mathbf{G}}^F$ and so $hgh^{-1} = \tilde{t}\tilde{t}^{-1}g\tilde{t}\tilde{t}^{-1} = tgt^{-1} =$

g' , that is, g and g' are conjugate in $\tilde{\mathbf{G}}^F$. Since ρ is unipotent, there exists some F -stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$ such that ρ occurs in $R_{\mathbf{T}}^{\mathbf{G}}(1)$ (where 1 stands for the trivial character of \mathbf{T}^F). There is an F -stable maximal torus $\tilde{\mathbf{T}} \subseteq \tilde{\mathbf{G}}$ such that $\mathbf{T} \subseteq \tilde{\mathbf{T}}$. Since $R_{\mathbf{T}}^{\mathbf{G}}(1)$ is the restriction of $R_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{G}}}(1)$ to \mathbf{G}^F (see [21, Remark 2.3.16]), there exists some $\tilde{\rho} \in \text{Unip}(\tilde{\mathbf{G}})$ such that ρ occurs in the restriction of $\tilde{\rho}$ to \mathbf{G}^F . But it is known that $\tilde{\rho}|_{\mathbf{G}^F}$ is irreducible (see [21, Lemma 2.3.14]) and so ρ is equal to the restriction of $\tilde{\rho}$. Thus, we certainly have $\rho(g) = \tilde{\rho}(g) = \tilde{\rho}(g') = \rho(g')$. \square

Example 4.5. Let \mathbf{G} be simple of type G_2 . In this case, the complete character table of \mathbf{G}^F is known; see Chang–Ree [5] ($p \neq 2, 3$), Enomoto [11] ($p = 3$) and Enomoto–Yamada [12] ($p = 2$). Now, there are four cuspidal character sheaves, and they are all unipotent; see [36, §20], [53, §6, §7]. From the known character tables, the above identities (\clubsuit') and the required scalars ζ can be easily extracted. For example, if $p \neq 2, 3$, then the four functions Y_1, Y_2, Y_3, Y_4 printed on [5, p. 411] are characteristic functions of the four cuspidal character sheaves on \mathbf{G} .

Let us go back to the general case. Implicit in (\clubsuit) and (\clubsuit') is the problem of choosing a convenient representative $g_1 \in \Sigma^F$. In a number of cases, Σ consists of regular elements in \mathbf{G} . In such a case, there are additional techniques to single out a canonical choice for $g_1 \in \Sigma^F$; see Corollary 4.8 below.

4.6. Regular elements. An element $g \in \mathbf{G}$ is called regular, if $\dim C_{\mathbf{G}}(g)$ is as small as possible; it is known that this is equivalent to the condition that $\dim C_{\mathbf{G}}(g) = \dim \mathbf{T}_0$. Furthermore, let $g = su = us$ be the Jordan decomposition of g (where s is semisimple and u is unipotent). Then g is regular if and only if u is regular in $C_{\mathbf{G}}^{\circ}(s)$. (For all this see, for example, [10, §12.1].) By Steinberg [59, Theorem 1.2], every semisimple element of \mathbf{G} is the semisimple part of some regular element; finally, two regular elements of \mathbf{G} are conjugate if and only if their semisimple parts are conjugate. In particular, all regular unipotent elements are conjugate.

Assume now that \mathbf{G} is simple and simply connected. Then a cross-section for the conjugacy classes of regular elements has been found by Steinberg [59]. Let us write $\mathbf{B} = \mathbf{U}\mathbf{T}_0$ where \mathbf{U} is the unipotent radical of \mathbf{B} . Let $\mathbf{B}^- \subseteq \mathbf{G}$ be the opposite Borel subgroup; then $\mathbf{B}^- = \mathbf{U}^-\mathbf{T}_0$, where \mathbf{U}^- is the unipotent radical of \mathbf{B}^- , and we have $\mathbf{U} \cap \mathbf{U}^- = \{1\}$. Let $w_c := w_{\alpha_1} \cdots w_{\alpha_r} \in \mathbf{W}$ be a Coxeter element, where $r = \dim \mathbf{T}_0$ and $\alpha_1, \dots, \alpha_r$ is a fixed enumeration of the simple roots in Φ^+ . Then the required cross-section is given by

$$\mathcal{N}_{w_c} := \mathbf{U}w_c \cap w_c\mathbf{U}^- \subseteq \mathbf{U}w_c\mathbf{U} \subseteq \mathbf{B}w_c\mathbf{B}.$$

Indeed, by [59, Theorem 1.4, Lemma 7.3], all elements of $\mathcal{N}_{\dot{w}_c}$ are regular and every regular element of \mathbf{G} is conjugate to exactly one element in $\mathcal{N}_{\dot{w}_c}$. And everything takes place inside the single double coset $\mathbf{U}\dot{w}_c\mathbf{U}$; note that this depends on the choice of the representative \dot{w}_c of $w_c \in \mathbf{W}$. The following result is a very special case of the results on “ C -small” classes in [43, §5], so we include the proof here. (It is already mentioned in the proof of [38, Lemma 8.10].)

Proposition 4.7 (Steinberg, He–Lusztig). *Assume that \mathbf{G} is simple and simply connected. Let Σ be a \mathbf{G} -conjugacy class of regular elements.*

- (a) *The set $\Sigma \cap \mathbf{U}\dot{w}_c\mathbf{U} \neq \emptyset$ is a single \mathbf{U} -orbit (under conjugation).*
- (b) *The set $\Sigma \cap \mathbf{B}\dot{w}_c\mathbf{B} \neq \emptyset$ is a single \mathbf{B} -orbit (under conjugation).*
- (c) *In (a), the stabilisers are trivial; in (b) they are equal to $Z(\mathbf{G})$.*

Proof. By He–Lusztig [24, Theorem 3.6(ii)], the map

$$\mathbf{U} \times \mathcal{N}_{\dot{w}_c} \rightarrow \mathbf{U}\dot{w}_c\mathbf{U}, \quad (u, z) \mapsto uzu^{-1},$$

is bijective. (A closely related result is stated in [59, Prop. 8.9], but since the proof is omitted there, we cite [24]; see also [2, §10].) Let us denote by g the unique element in $\Sigma \cap \mathcal{N}_{\dot{w}_c}$; in particular, $g \in \Sigma \cap \mathbf{U}\dot{w}_c\mathbf{U}$.

(a) Given any $g' \in \Sigma \cap \mathbf{U}\dot{w}_c\mathbf{U}$, we can write $g' = uzu^{-1}$ where $u \in \mathbf{U}$ and $z \in \mathcal{N}_{\dot{w}_c}$. Thus, the two elements z and g in $\mathcal{N}_{\dot{w}_c}$ are conjugate in \mathbf{G} . But then we must have $z = g$ and so g' is conjugate to g under \mathbf{U} .

(b) Take any $g' \in \Sigma \cap \mathbf{B}\dot{w}_c\mathbf{B}$. Since $\mathbf{B}\dot{w}_c\mathbf{B} = \mathbf{U}\mathbf{T}_0\dot{w}_c\mathbf{U}$, we can write $g' = u_1t\dot{w}_cu_2$ where $u_1, u_2 \in \mathbf{U}$ and $t \in \mathbf{T}_0$. By [59, Lemma 7.6], we can further write $t\dot{w}_c = \tilde{t}\dot{w}_c\tilde{t}^{-1}$ for some $\tilde{t} \in \mathbf{T}_0$. Then

$$\tilde{t}^{-1}g'\tilde{t} = \tilde{t}^{-1}u_1t\dot{w}_cu_2\tilde{t} = (\tilde{t}^{-1}u_1\tilde{t})\dot{w}_c(\tilde{t}^{-1}u_2\tilde{t}) \in \mathbf{U}\dot{w}_c\mathbf{U}.$$

So we have $\tilde{t}^{-1}g'\tilde{t} = uzu^{-1}$ where $u \in \mathbf{U}$ and $z \in \mathcal{N}_{\dot{w}_c}$. Thus, $z \in \mathcal{N}_{\dot{w}_c}$ and $g \in \mathcal{N}_{\dot{w}_c}$ are conjugate in \mathbf{G} and so $z = g$. Hence, g' is conjugate to g under \mathbf{B} .

(c) The bijectivity of the above map $\mathbf{U} \times \mathcal{N}_{\dot{w}_c} \rightarrow \mathbf{U}\dot{w}_c\mathbf{U}$ immediately implies that $C_{\mathbf{U}}(g) = \{1\}$; thus, the stabilisers are trivial in (a). For (b), we must show that $\text{Stab}_{\mathbf{B}}(g) = Z(\mathbf{G})$. So let $b \in \mathbf{B}$ be such that $bgb^{-1} = g$. Writing $g = v\dot{w}_c$ (where $v \in \mathbf{U}$) and $b = ut$ (where $u \in \mathbf{U}$ and $t \in \mathbf{T}_0$), we obtain

$$v\dot{w}_c = g = bgb^{-1} = utv\dot{w}_ct^{-1}u^{-1} = (utvt^{-1})\dot{w}_c(\dot{w}_c^{-1}t\dot{w}_ct^{-1})u^{-1}.$$

Setting $\tilde{t} := \dot{w}_c^{-1}t\dot{w}_ct^{-1} \in \mathbf{T}_0$, we see that the left hand side lies in the double coset $\mathbf{U}\dot{w}_c\mathbf{U}$, and the right hand side lies in the double coset $\mathbf{U}\dot{w}_c\tilde{t}\mathbf{U}$. But

then the sharp form of the Bruhat decomposition implies that $\tilde{t} = 1$ and so $t = \dot{w}_c^{-1}t\dot{w}_c$. By [59, Remarks 7.7(b)], this forces $t \in Z(\mathbf{G})$. But then $u \in C_{\mathbf{U}}(g)$ and so $u = 1$. Hence, $\text{Stab}_{\mathbf{B}}(g) \subseteq Z(\mathbf{G})$; the reverse inclusion is clear. \square

Corollary 4.8. *In the setting of Proposition 4.7, assume that Σ is F -stable and $F(\dot{w}_c) = \dot{w}_c$. Then there exists a unique \mathbf{G}^F -conjugacy class $C \subseteq \Sigma^F$ such that $C \cap \mathbf{U}^F \dot{w}_c \mathbf{U}^F \neq \emptyset$. Furthermore, we have $C \cap \mathcal{N}_{\dot{w}_c} \neq \emptyset$.*

Proof. By Proposition 4.7, the group \mathbf{U} acts transitively on $X := \Sigma \cap \mathbf{U} \dot{w}_c \mathbf{U}$ by conjugation, and we have $\text{Stab}_{\mathbf{U}}(x) = \{1\}$ for all $x \in X$; in particular, $\text{Stab}_{\mathbf{U}}(x)$ is connected. A standard application of Lang’s Theorem (see, e.g., [21, Prop. 1.4.9]) shows that $X^F \neq \emptyset$ and that X^F is a single \mathbf{U}^F -orbit. Thus, there exists a unique \mathbf{G}^F -conjugacy class $C \subseteq \Sigma^F$ such that $C \cap (\mathbf{U} \dot{w}_c \mathbf{U})^F \neq \emptyset$. Note that $(\mathbf{U} \dot{w}_c \mathbf{U})^F = \mathbf{U}^F \dot{w}_c \mathbf{U}^F$, by the sharp form of the Bruhat decomposition.

Finally note that, since $F(\dot{w}_c) = \dot{w}_c$, we have $F(\mathcal{N}_{\dot{w}_c}) = \mathcal{N}_{\dot{w}_c}$. Let g be the unique element in $\Sigma \cap \mathcal{N}_{\dot{w}_c}$. Then we also have $F(g) \in \Sigma \cap \mathcal{N}_{\dot{w}_c}$ and so $F(g) = g$. Hence, $g \in \Sigma^F$ and $g \in \mathbf{U}^F \dot{w}_c \mathbf{U}^F$. So C must be the \mathbf{G}^F -conjugacy class of g . \square

Example 4.9. Let \mathbf{G}, Σ, C be as in Corollary 4.8. Then, clearly, we also have $C \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F \neq \emptyset$. Since the stabilisers for the action of \mathbf{B} on $\Sigma \cap \mathbf{B} \dot{w}_c \mathbf{B}$ are equal to $Z(\mathbf{G})$, the set $(\Sigma \cap \mathbf{B} \dot{w}_c \mathbf{B})^F$ will split into finitely many \mathbf{B}^F -orbits, indexed by the F -conjugacy classes of $Z(\mathbf{G})$. More precisely, let g be the unique element in $C \cap \mathcal{N}_{\dot{w}_c}$. Let $z_1, \dots, z_r \in Z(\mathbf{G})$ be representatives of the F -conjugacy classes of $Z(\mathbf{G})$, where $z_1 = 1$. For $1 \leq i \leq r$, we set $g_i := t_i g t_i^{-1}$, where $t_i \in \mathbf{T}_0$ is such that $z_i = t_i^{-1} F(t_i)$; here, we also assume $t_1 = 1$. Then g_1, \dots, g_r are representatives of the \mathbf{B}^F -orbits on $(\Sigma \cap \mathbf{B} \dot{w}_c \mathbf{B})^F$ (see [58, I, 2.7]). Hence,

$$(a) \quad (\Sigma \cap \mathbf{B} \dot{w}_c \mathbf{B})^F = (C_1 \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F) \cup \dots \cup (C_r \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F),$$

where C_i is the \mathbf{G}^F -conjugacy class of g_i , for all i . Since $z_i = t_i^{-1} F(t_i) \in Z(\mathbf{G})$, the map $C_{\mathbf{G}}(g)^F \rightarrow C_{\mathbf{G}}(g_i)^F, x \mapsto t_i x t_i^{-1}$, is bijective. In particular, we have

$$(b) \quad |C| = |C_i| \quad \text{for } 1 \leq i \leq r.$$

The union in (a) may not be disjoint; but $C_i = C_j$ can only happen if the images of z_i and z_j in $A_{\mathbf{G}}(g)$ are F -conjugate. Now, it is known that $A_{\mathbf{G}}(g)$ is abelian, see [58, III, 1.16 and 1.17]; so, for example, if the natural map $Z(\mathbf{G}) \rightarrow A_{\mathbf{G}}(g)$ is injective and F acts trivially on $A_{\mathbf{G}}(g)$, then the C_i are all

distinct. (This will cover most examples that we consider.) Finally, we note the implication:

$$(c) \quad Z(\mathbf{G})^F = \{1\} \quad \Rightarrow \quad (\Sigma \cap \mathbf{B}\dot{w}_c\mathbf{B})^F = C \cap \mathbf{B}^F\dot{w}_c\mathbf{B}^F.$$

Indeed, if $Z(\mathbf{G})^F = \{1\}$, then all elements of $Z(\mathbf{G})$ are F -conjugate and so $r = 1$.

Lemma 4.10. *Let \mathbf{G} , Σ , C be as in Corollary 4.8. Assume, furthermore, that \mathbf{G} is of split type, that $\Sigma = \Sigma^{-1}$, and that the union in Example 4.9(a) is disjoint.*

- (a) *There is a permutation $i \mapsto i'$ (of order 2) of the set $\{1, \dots, r\}$ such that $C_i^{-1} = C_{i'}$ for $1 \leq i \leq r$.*
- (b) *If r is odd (e.g., if $Z(\mathbf{G})^F = \{1\}$), then we have $C_i = C_i^{-1}$ for $1 \leq i \leq r$.*

Proof. First note that $C_i^{-1} \subseteq \Sigma^{-1} = \Sigma$ for $1 \leq i \leq r$. We claim that

$$C_i^{-1} \cap \mathbf{B}^F\dot{w}_c\mathbf{B}^F \neq \emptyset \quad \text{for all } i.$$

To see this, it is enough to show that $C_i \cap \mathbf{B}^F\dot{w}_c^{-1}\mathbf{B}^F \neq \emptyset$. Now note that w_c^{-1} also is a Coxeter element (of minimal length); it is well-known that w_c and w_c^{-1} are conjugate in \mathbf{W} . By [43, 0.2], [16, Cor. 3.7(a)], we have

$$|C_i \cap \mathbf{B}^F\dot{w}\mathbf{B}^F| = |C_i \cap \mathbf{B}^F\dot{w}'\mathbf{B}^F|$$

for any two elements $w, w' \in \mathbf{W}$ that are conjugate in \mathbf{W} and of minimal length in their conjugacy class. In particular, we can conclude that

$$|C_i^{-1} \cap \mathbf{B}^F\dot{w}_c\mathbf{B}^F| = |C_i \cap \mathbf{B}^F\dot{w}_c^{-1}\mathbf{B}^F| = |C_i \cap \mathbf{B}^F\dot{w}_c\mathbf{B}^F| \neq 0$$

for all i , as required. Thus, the above claim is proved.

(a) Let $i \in \{1, \dots, r\}$. By the above claim, C_i^{-1} is a \mathbf{G}^F -conjugacy class that is contained in $(\Sigma \cap \mathbf{B}\dot{w}_c\mathbf{B})^F$. So, by Example 4.9(a), we must have $C_i^{-1} = C_{i'}$ for some $i' \in \{1, \dots, r\}$. Since the C_i are all distinct, i' is uniquely determined by i ; furthermore, the map $i \mapsto i'$ is a permutation (of order 2) of the set $\{1, \dots, r\}$.

(b) Assume that r is odd. Then there must be some $i_0 \in \{1, \dots, r\}$ such that $i_0 = i'_0$, that is, $C_{i_0} = C_{i_0}^{-1}$. Now recall that C_{i_0} is the \mathbf{G}^F -conjugacy class of g_{i_0} , where $g_{i_0} = t_{i_0}gt_{i_0}^{-1}$ and $t_{i_0} \in \mathbf{T}_0$ is such that $z_{i_0} = t_{i_0}^{-1}F(t_{i_0})$. There exists some $x \in \mathbf{G}^F$ such that $g_{i_0}^{-1} = xg_{i_0}x^{-1}$. It follows that $g^{-1} = ygy^{-1}$, where $y := t_{i_0}^{-1}xt_{i_0}$. Now $F(y) = F(t_{i_0})^{-1}xF(t_{i_0}) = z_{i_0}^{-1}t_{i_0}^{-1}xt_{i_0}z_{i_0} = y$, since $z_{i_0} \in Z(\mathbf{G})$. Thus, we have shown that $C = C^{-1}$. Then the same argument, applied to any $i \in \{1, \dots, r\}$, also yields that $C_i = C_i^{-1}$. □

The following example (pointed out to the author by G. Malle) shows that the situation is really different when r is even.

Example 4.11. Let $\mathbf{G} = \mathrm{SL}_2(k)$ with $p \neq 2$. Then $Z(\mathbf{G})$ has order 2 and so $r = 2$. Let Σ be the class of regular unipotent elements; we certainly have $\Sigma = \Sigma^{-1}$. Let \mathbf{B} be the Borel subgroup consisting of the upper triangular matrices in \mathbf{G} ; also let $\mathbf{W} = \langle s_1 \rangle$. Then one checks that

$$g := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \Sigma^F \cap \mathbf{U} \dot{s}_1 \mathbf{U}^F \quad \text{where} \quad \dot{s}_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

So the unique class C in Corollary 4.8 is the \mathbf{G}^F -conjugacy class of g . Now Σ^F splits into two classes in \mathbf{G}^F , with representatives g and

$$g' := \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \in \Sigma^F, \quad \text{where } \xi \in \mathbb{F}_q^\times \text{ is not a square in } \mathbb{F}_q^\times.$$

(One checks that, indeed, $g' \in \mathbf{B}^F \dot{s}_1 \mathbf{B}^F$ but $g' \notin \mathbf{U}^F \dot{s}_1 \mathbf{U}^F$.) Furthermore, one checks that g and g^{-1} are conjugate in \mathbf{G}^F if and only if -1 is a square in \mathbb{F}_q^\times , that is, if and only if $q \equiv 1 \pmod{4}$. Hence, if $q \equiv 3 \pmod{4}$, then $C \neq C^{-1}$.

Remark 4.12. Assume that \mathbf{G} is of split type (then $\sigma = \mathrm{id}_{\mathbf{W}}$). Let Σ be an arbitrary F -stable conjugacy class of \mathbf{G} and $w \in \mathbf{W}$. For any $g \in \Sigma^F$ denote by C_g the \mathbf{G}^F -conjugacy class of g . Then the cardinalities $|C_g \cap \mathbf{B}^F \dot{w} \mathbf{B}^F|$ can be computed using the representation theory of \mathbf{G}^F ; see, e.g., [43, 1.2(a)]. For this purpose, we consider the induced character $\mathrm{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}(1)$ (where 1 stands for the trivial character of \mathbf{B}^F) and let \mathcal{H}_q be the corresponding Hecke algebra, that is, the endomorphism algebra of a $\mathbb{K}\mathbf{G}^F$ -module affording $\mathrm{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}(1)$. This algebra has a standard basis usually denoted by $\{T_w \mid w \in \mathbf{W}\}$, where

$$T_{w_\alpha}^2 = qT_1 + (q - 1)T_{w_\alpha} \quad \text{for every simple root } \alpha \in \Phi.$$

There is a bijection, $\phi \leftrightarrow \phi^{(q)}$, between $\mathrm{Irr}(\mathbf{W})$ and the irreducible characters of \mathcal{H}_q (which is canonical once a square root $\sqrt{q} \in \mathbb{K}$ has been fixed; see, e.g., [23, §9.3].) Via this correspondence, the irreducible characters of \mathbf{G}^F that occur in $\mathrm{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}(1)$ are parametrised by $\mathrm{Irr}(\mathbf{W})$ (see, e.g., [23, §8.4]); we denote by $[\phi] \in \mathrm{Irr}(\mathbf{G}^F)$ the character corresponding to $\phi \in \mathrm{Irr}(\mathbf{W})$. Then we have:

$$(a) \quad |C_g \cap \mathbf{B}^F \dot{w} \mathbf{B}^F| = \frac{|\mathbf{B}^F|}{|C_{\mathbf{G}}(g)^F|} \sum_{\phi \in \mathrm{Irr}(\mathbf{W})} \phi^{(q)}(T_w) [\phi](g)$$

for any $w \in \mathbf{W}$ and any $g \in \mathbf{G}^F$. (See [43, 1.2(a)] and [16, Remark 3.6] for further details and references.) The values $\phi^{(a)}(T_w)$ are explicitly known (or there are explicit combinatorial algorithms); see [23]. Hence, if we have sufficient information on the values of $[\phi] \in \text{Irr}(\mathbf{G}^F)$, then we can work out the cardinality $|C_g \cap \mathbf{B}^F \dot{w} \mathbf{B}^F|$, and this will be useful to identify $C_g \subseteq \Sigma^F$.

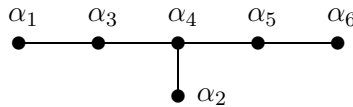
Assume now that we are in the setting of §4.6, where \mathbf{G} is simple and simply connected, Σ consists of regular elements and $w = w_c$ is a Coxeter element. Also assume that the natural map $Z(\mathbf{G}) \rightarrow A_{\mathbf{G}}(g_1)$ is injective and F acts trivially on $A_{\mathbf{G}}(g_1)$. Let C_1, \dots, C_r be the \mathbf{G}^F -conjugacy classes that are contained in Σ^F and have a non-empty intersection with $\mathbf{B}^F \dot{w}_c \mathbf{B}^F$. Then, by Example 4.9, we have $r = |Z(\mathbf{G})|$ and each set $C_i \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F$ is a single \mathbf{B}^F -orbit, of size $|\mathbf{B}^F|/r$. Hence, the above identity (a) can be expressed as follows.

$$(b) \quad \sum_{\phi \in \text{Irr}(\mathbf{W})} \phi^{(a)}(T_w) [\phi](g) = \begin{cases} \frac{1}{r} |C_{\mathbf{G}}(g)^F| & \text{if } g \in C_1 \cup \dots \cup C_r, \\ 0 & \text{if } g \in \Sigma^F \setminus (C_1 \cup \dots \cup C_r). \end{cases}$$

This identity can be exploited to obtain information on the character values $[\phi](g)$ and, hence, potentially, on the unknown scalars ζ in §4.3(♣'); see the proof of Proposition 6.5 below for an example.

5. Cuspidal unipotent character sheaves in type E_6

Throughout this section, let \mathbf{G} be simple, simply connected of type E_6 . Let $q = p^f$ (where $f \geq 1$) be such that $F: \mathbf{G} \rightarrow \mathbf{G}$ defines an \mathbb{F}_q -rational structure. Except for §5.6 (at the very end), we assume that \mathbf{G} is of split type; thus, $\sigma = \text{id}_{\mathbf{W}}$ and the permutation $\alpha \mapsto \alpha^\dagger$ of Φ is the identity. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ be the set of simple roots in Φ^+ , where the labelling is chosen as follows.



If $p = 3$, then the cuspidal character sheaves and almost characters have been considered by Hetz [25]. So assume from now on that $p \neq 3$. Let $\alpha_0 \in \Phi$ be the unique root of maximal height and consider the subsystem $\Phi_0 \subseteq \Phi$ of type $A_2 \times A_2 \times A_2$ spanned by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_0\}$. The relevance of this particular example is that Φ_0 occurs in the classification of cuspidal unipotent character sheaves on \mathbf{G} ; see [36, Prop. 20.3] (and also the remarks in [54, 5.2]).

Using CHEVIE, we find that the unique set Δ_0 of simple roots in $\Phi_0 \cap \Phi^+$ is given by

$$\Delta_0 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha'_0 := \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \}.$$

(In the Dynkin diagram of Δ_0 , the node α'_0 is joined to α_2 .) There are three equivalence classes of pairs $(\Phi', w) \in \Xi$ under \sim , where $\Phi' = \Phi_0$; representatives (Φ_0, d_i) , where $d_i \in \mathbf{W}$ has minimal length in $\mathbf{W}(\Phi_0)d_i$ for $i = 1, 2, 3$, are given in Table 3. In that table, an expression like 4315... means the product $w_{\alpha_4}w_{\alpha_3}w_{\alpha_1}w_{\alpha_5} \cdots$ in \mathbf{W} . Otherwise, the conventions are the same as in Example 2.6. In particular, recall that the permutation in the second column refers to the simple roots in Δ_0 , as listed above, not to those in Δ . (So, e.g., the cycle $(3, 4, 6)$ means that $\alpha_3 \mapsto \alpha_5 \mapsto \alpha'_0 \mapsto \alpha_3$.)

Table 3: The subsystem Φ_0 for type E_6

d_i	permutation	σ'_i -classes
$d_1 = 1_{\mathbf{W}}$	$()$	27
$d_2 = 431543654$	$(1, 4)(2, 6)(3, 5)$	9
$d_3 = 425431654234$	$(1, 5, 2)(3, 4, 6)$	3

5.1. The subgroup $H' = \langle T_0, U_\alpha (\alpha \in \Phi_0) \rangle$. For each root $\alpha \in \Phi$, denote by $\alpha^\vee: k^\times \rightarrow T_0$ the corresponding coroot. Since \mathbf{G} is simply connected, every $t \in T_0$ has a unique expression $t = h(\xi_1, \dots, \xi_6) := \prod_{1 \leq i \leq 6} \alpha_i^\vee(\xi_i)$ where $\xi_i \in k^\times$ for $1 \leq i \leq 6$. By [21, Example 1.5.6], we have

$$Z(\mathbf{G}) = \{h(\xi, 1, \xi^{-1}, 1, \xi, \xi^{-1}) \mid \xi \in k^\times, \xi^3 = 1\}.$$

A similar computation shows that

$$Z(\mathbf{H}') = \{h(\xi, 1, \xi^{-1}, 1, \zeta, \zeta^{-1}) \mid \xi, \zeta \in k^\times, \xi^3 = \zeta^3 = 1\}.$$

Thus, $Z(\mathbf{H}')$ is generated by $Z(\mathbf{G})$ and any fixed $t \in Z(\mathbf{H}') \setminus Z(\mathbf{G})$. Since q is not a power of 3, we have $|Z(\mathbf{G})| = 3$ and $|Z(\mathbf{H}')| = 9$. Given $t = h(\xi, 1, \xi^{-1}, 1, \zeta, \zeta^{-1}) \in Z(\mathbf{H}')$ (where $\xi^3 = \zeta^3 = 1$), we have $C_{\mathbf{G}}(t) = \mathbf{H}'$ if and only if $\xi \neq \zeta$. Furthermore, one easily checks that

$$\begin{aligned} \dot{d}_2^{-1}t\dot{d}_2 &= z_2(t)t^{-1} & \text{where } z_2(t) &:= h(\xi\zeta, 1, (\xi\zeta)^{-1}, 1, \xi\zeta, (\xi\zeta)^{-1}) \in Z(\mathbf{G}), \\ \dot{d}_3^{-1}t\dot{d}_3 &= z_3(t)t & \text{where } z_3(t) &:= h(\xi\zeta^{-1}, 1, \xi^{-1}\zeta, 1, \xi\zeta^{-1}, \xi^{-1}\zeta) \in Z(\mathbf{G}). \end{aligned}$$

These two relations show that all elements in $Z(\mathbf{H}') \setminus Z(\mathbf{G})$ are conjugate in \mathbf{G} . In particular, if we choose $\zeta = \xi^{-1} \neq 1$, then $d_2^{-1}td_2 = t^{-1}$. Thus, if \mathcal{C} denotes the \mathbf{G} -conjugacy class of the elements in $Z(\mathbf{H}') \setminus Z(\mathbf{G})$, then

$$F(\mathcal{C}) = \mathcal{C}, \quad \mathcal{C} = \mathcal{C}^{-1} \quad \text{and} \quad \mathcal{C} = Z(\mathbf{G})\mathcal{C}.$$

In order to see that \mathcal{C} is F -stable, we argue as follows. If $q \equiv 1 \pmod 3$, then $F(z) = z$ for all $z \in Z(\mathbf{H}')$. On the other hand, if $q \equiv 2 \pmod 3$, then $F(z) = z^{-1}$ for all $z \in Z(\mathbf{H}')$ and, hence, $Z(\mathbf{H}')^F = \{1\}$. But, if $t = h(\xi, 1, \xi^{-1}, 1, \zeta, \zeta^{-1}) \in \mathcal{C}$ with $\zeta = \xi^{-1} \neq 1$ as above, then $F(t) = t^{-1} = d_2^{-1}td_2 \in \mathcal{C}$, as required.

5.2. The conjugacy class $\Sigma \subseteq \mathbf{G}$. Let us fix an element $s_1 \in \mathcal{C}$; since \mathcal{C} is F -stable, we may assume that $F(s_1) = s_1$. Now $\mathbf{H}'_1 := C_{\mathbf{G}}(s_1)$ is conjugate to \mathbf{H}' in \mathbf{G} . Let $u_1 \in \mathbf{H}'_1$ be regular unipotent; since all regular unipotent elements in \mathbf{H}'_1 are conjugate in \mathbf{H}'_1 , we may assume that $F(u_1) = u_1$. Let Σ be the \mathbf{G} -conjugacy class of $g_1 := s_1u_1$; then Σ is F -stable since $F(g_1) = g_1$. Since $Z(\mathbf{G})\mathcal{C} = \mathcal{C} = \mathcal{C}^{-1}$, one also deduces that $Z(\mathbf{G})\Sigma = \Sigma = \Sigma^{-1}$. We claim that

$$A_{\mathbf{G}}(g_1) \text{ is generated by } \bar{s}_1 \text{ and all } \bar{z} \text{ for } z \in Z(\mathbf{G});$$

here, for any $c \in C_{\mathbf{G}}(g_1)$, we denote by \bar{c} the image of c in $A_{\mathbf{G}}(g_1)$. Indeed, we have $C_{\mathbf{G}}(g_1) = C_{\mathbf{H}'_1}(u_1)$ and so $C_{\mathbf{G}}^{\circ}(g_1) = C_{\mathbf{H}'_1}^{\circ}(u_1)$. Since we are in good characteristic (inside $\mathbf{H}'_1 \cong \mathbf{H}'$, which is of type $A_2 \times A_2 \times A_2$) and $Z(\mathbf{H}'_1)$ is finite, it follows that $A_{\mathbf{G}}(g_1) = A_{\mathbf{H}'_1}(u_1) \cong Z(\mathbf{H}'_1)$, where the isomorphism is induced by the natural map $Z(\mathbf{H}'_1) \subseteq C_{\mathbf{H}'_1}(u_1) \rightarrow A_{\mathbf{H}'_1}(u_1)$; see [10, §12.3]. Note also that $\bar{g}_1 = \bar{s}_1$. By §5.1, we have that $Z(\mathbf{H}')$ is generated by $Z(\mathbf{G})$ and any fixed element in $Z(\mathbf{H}') \setminus Z(\mathbf{G})$. Consequently, $Z(\mathbf{H}'_1)$ is generated by $Z(\mathbf{G})$ and s_1 , which implies the above claim.

Note that F acts trivially on $A_{\mathbf{G}}(g_1)$ if $q \equiv 1 \pmod 3$. On the other hand, if $q \equiv 2 \pmod 3$, then $F(z) = z^{-1}$ for all $z \in Z(\mathbf{G})$ and, hence, F acts non-trivially on $A_{\mathbf{G}}(g_1)$; in this case, we have $A_{\mathbf{G}}(g_1)^F = \langle \bar{s}_1 \rangle \cong \mathbb{Z}/3\mathbb{Z}$. Hence, the set Σ^F splits into an odd number (either 9 or 3) of conjugacy classes in \mathbf{G}^F . So, among these classes, there must be at least one that is equal to its inverse; we now choose $g_1 \in \Sigma^F$ to be in such a class; thus, g_1 is conjugate to g_1^{-1} in \mathbf{G}^F (and not just in \mathbf{G}). Note also that, using Lemma 4.10(b), we could fix the \mathbf{G}^F -conjugacy class of g_1 completely, by requiring that $g_1 \in C$, where C is the \mathbf{G}^F -conjugacy class determined by Σ (and the choice of w_c) as in Corollary 4.8.

5.3. Cuspidal unipotent character sheaves. First we consider the group $\tilde{\mathbf{G}} := \mathbf{G}/Z(\mathbf{G})$. Let $\pi: \mathbf{G} \rightarrow \tilde{\mathbf{G}}$ be the canonical map; let $\tilde{\Sigma} := \pi(\Sigma)$ and

$\tilde{g}_1 := \pi(g_1)$. By the proof of [36, Prop. 20.3(a)] (see also [54, 4.6, 5.2]), there are two cuspidal unipotent character sheaves \tilde{A}_1 and \tilde{A}_2 of $\tilde{\mathbf{G}}$; they have support $\tilde{\Sigma}$ and they are F -invariant. As explained in the proof of [36, Cor. 20.4] (see also [54, p. 347]), the local systems (see §4.2) associated with \tilde{A}_1 and \tilde{A}_2 are one-dimensional; they correspond to linear characters $\tilde{\psi}_1$ and $\tilde{\psi}_2$ of $A_{\tilde{\mathbf{G}}}(\tilde{g}_1)$, such that $\tilde{\psi}_1(\tilde{g}_1) = \theta$ and $\tilde{\psi}_2(\tilde{g}_1) = \theta^2$, where $1 \neq \theta \in \mathbb{K}^\times$ is a fixed third root of unity. (Here, \tilde{g}_1 denotes the image of \tilde{g}_1 in $A_{\tilde{\mathbf{G}}}(\tilde{g}_1)$.) Now use π to go back to \mathbf{G} . First note that π canonically induces a group homomorphism $\bar{\pi}: A_{\mathbf{G}}(g_1) \rightarrow A_{\tilde{\mathbf{G}}}(\tilde{g}_1)$. Hence, we obtain irreducible characters of $A_{\mathbf{G}}(g_1)$ by setting

$$\psi_1 := \tilde{\psi}_1 \circ \bar{\pi} \in \text{Irr}(A_{\mathbf{G}}(g_1)) \quad \text{and} \quad \psi_2 := \tilde{\psi}_2 \circ \bar{\pi} \in \text{Irr}(A_{\mathbf{G}}(g_1)).$$

By the proof of [36, Prop. 20.3(b)], $A_1 := \pi^* \tilde{A}_1$ and $A_2 := \pi^* \tilde{A}_2$ are cuspidal unipotent character sheaves on \mathbf{G} (and these are the only ones); they have support Σ and they correspond to the linear characters ψ_1 and ψ_2 of $A_{\mathbf{G}}(g_1)$. (See the general reduction techniques described in [34, 2.10].) Finally note that, clearly, the image of $Z(\mathbf{G})$ in $A_{\mathbf{G}}(g_1)$ is contained in $\ker(\bar{\pi})$. Hence, we have

$$\psi_1(\bar{s}_1) = \theta, \quad \psi_2(\bar{s}_1) = \theta^2, \quad \psi_1(\bar{z}) = \psi_2(\bar{z}) = 1 \quad \text{for } z \in Z(\mathbf{H}'_1)^F.$$

(Recall that $\bar{g}_1 = \bar{s}_1$.) Thus, ψ_1 and ψ_2 are completely determined, where ψ_2 is the complex conjugate of ψ_1 . Furthermore, the roots of unity attached to A_1 and A_2 as in §4.2 are $\lambda_{A_1} = \theta$ and $\lambda_{A_2} = \theta^2$. Using ψ_1 and ψ_2 , we can now write down characteristic functions of A_1 and A_2 , as in §4.2; we have $\chi_{g_1, \psi_i} = \chi_{\tilde{g}_1, \tilde{\psi}_i} \circ \pi$ for $i = 1, 2$. (Recall that $\Sigma = Z(\mathbf{G})\Sigma$ and so $\Sigma = \pi^{-1}(\tilde{\Sigma})$.)

5.4. Unipotent characters and almost characters. Let $\tilde{\mathbf{G}} = \mathbf{G}/Z(\mathbf{G})$ and $\pi: \mathbf{G} \rightarrow \tilde{\mathbf{G}}$ as above. First note that the unipotent characters of \mathbf{G}^F and $\tilde{\mathbf{G}}^F$ can be canonically identified via π (see, e.g., [21, Prop. 2.3.15]). As discussed in §4.3, the unipotent characters are parametrised by a certain set $X(\mathbf{W})$ (which only depends on \mathbf{W}). This set is further partitioned into “families”; the interesting family for us is the one which contains $x \in X(\mathbf{W})$ such that $\rho_x \in \text{Unip}(\mathbf{G}^F)$ is cuspidal; it is given as follows (see the tables in [4, p. 480] and [33, p. 363]).

$x \in X(\mathbf{W}) :$	$(1, 1)$	$(1, \varepsilon)$	$(g_2, 1)$	$(g_3, 1)$	$(1, r)$	(g_2, ε)	(g_3, θ)	(g_3, θ^2)
$\rho_x \in \text{Unip}(\mathbf{G}^F) :$	$[80_s]$	$[20_s]$	$[60_s]$	$[10_s]$	$[90_s]$	$D_4[r]$	$E_6[\theta]$	$E_6[\theta^2]$

Here, labels such as $80_s, 20_s$ etc. denote irreducible characters of \mathbf{W} (as in [33, Chap. 4]); then $[80_s], [20_s]$ etc. are the corresponding irreducible constituents

of $\text{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}(1)$ where 1 stands for the trivial character of \mathbf{B}^F ; the characters $E_6[\theta]$ and $E_6[\theta^2]$ are cuspidal unipotent; $D_4[r]$ is a further non-cuspidal character. (Note also that, for example, the “ g_3 ” in (g_3, θ) and (g_3, θ^2) has nothing to do with elements in \mathbf{G}^F ; these are just notations for parameters in $X(\mathbf{W})$.) For each $x \in X(\mathbf{W})$ we also have a unipotent almost character R_x , formed using the entries of the corresponding Fourier matrix (the relevant part of which is printed in [4, p. 457]). In particular, we have:

$$\begin{aligned} R_{(g_3, \theta)} &:= \frac{1}{3}([80_s] + [20_s] - [10_s] - [90_s] + 2E_6[\theta] - E_6[\theta^2]), \\ R_{(g_3, \theta^2)} &:= \frac{1}{3}([80_s] + [20_s] - [10_s] - [90_s] - E_6[\theta] + 2E_6[\theta^2]). \end{aligned}$$

Now consider the two cuspidal unipotent character sheaves A_1 and A_2 described above, with characteristic functions χ_{g_1, ψ_1} and χ_{g_1, ψ_2} . By the main result of [54, §4] (see also [54, 5.2]), there are scalars $\zeta, \zeta' \in \mathbb{K}$ of absolute value 1 such that

$$R_{(g_3, \theta)} = \zeta \chi_{g_1, \psi_1} \quad \text{and} \quad R_{(g_3, \theta^2)} = \zeta' \chi_{g_1, \psi_2}.$$

(More precisely, in [54, §4], this is proved for $\tilde{\mathbf{G}}$ but the discussion in §5.3 shows that this also holds for \mathbf{G} , with ψ_1 and ψ_2 as above.) By [15, Table 1], the characters $E_6[\theta]$ and $E_6[\theta^2]$ are complex conjugate to each other, and their values lie in the field $\mathbb{Q}(\theta)$; furthermore, all characters $[\phi]$ (where $\phi \in \text{Irr}(\mathbf{W})$) are rational-valued; see [15, Prop. 5.6]. We conclude that the class functions $R_{(g_3, \theta)}$ and $R_{(g_3, \theta^2)}$ are complex conjugate to each other, and their values lie in $\mathbb{Q}(\theta)$. Now, since $\dim \mathbf{G} - \dim \Sigma = \dim \mathbf{T}_0 = 6$, we have

$$\begin{aligned} R_{(g_3, \theta)}(g_1) &= \zeta \chi_{g_1, \psi_1}(g_1) = \zeta q^3 \in \mathbb{Q}(\theta), \\ R_{(g_3, \theta^2)}(g_1) &= \zeta' \chi_{g_1, \psi_2}(g_1) = \zeta' q^3 \in \mathbb{Q}(\theta). \end{aligned}$$

Thus, we can already conclude that $\zeta' = \bar{\zeta}$. Since g_1 is conjugate to g_1^{-1} in \mathbf{G}^F , we have $E_6[\theta](g_1) = E_6[\theta](g_1^{-1}) = \overline{E_6[\theta]}(g_1) = E_6[\theta^2](g_1)$. Consequently, we also have $R_{(g_3, \theta)}(g_1) = R_{(g_3, \theta^2)}(g_1)$ and so $\zeta = \zeta' = \bar{\zeta}$. Since $\zeta \in \mathbb{Q}(\theta)$, this implies that $\zeta \in \mathbb{Q}$. And since ζ has absolute value 1, we must have $\zeta = \zeta' = \pm 1$.

Proposition 5.5. *In the above setting, recall that $g_1 \in \Sigma^F$ is conjugate to g_1^{-1} in \mathbf{G}^F . Then we have $\zeta = \zeta' = 1$, that is, $R_{(g_3, \theta)} = \chi_{g_1, \psi_1}$ and $R_{(g_3, \theta^2)} = \chi_{g_1, \psi_2}$.*

Proof. (See also [18, Remark 3.3].) Inverting the matrix relating unipotent characters and unipotent almost characters, we obtain:

$$E_6[\theta] = \frac{1}{3}(R_{80_s} + R_{20_s} - R_{10_s} - R_{90_s} + 2R_{(g_3, \theta)} - R_{(g_3, \theta^2)}).$$

Using the formula for R_ϕ in §4.3, the (known) character table of \mathbf{W} and Example 3.3, we find that

$$R_{80_s}(g_1) = R_{20_s}(g_1) = R_{90_s}(g_1) = 0 \quad \text{and} \quad R_{10_s}(g_1) = \epsilon,$$

where $\epsilon = \pm 1$ is such that $q \equiv \epsilon \pmod 3$. This yields $E_6[\theta](g_1) = \frac{1}{3}(-\epsilon + \zeta q^3) \in \mathbb{Q}$, where the left hand side is an algebraic integer. Hence, 3 must divide $\zeta q^3 - \epsilon \in \mathbb{Z}$. Since $\zeta = \pm 1$, the only possibility is that $\zeta = 1$. \square

Table 4: Values of $E_6[\theta]$ on Σ^F

$q \equiv 1 \pmod 3$	$\{\bar{1}, \bar{z}, \bar{z}^2\}$	$\{\bar{s}_1, \bar{s}_1\bar{z}, \bar{s}_1\bar{z}^2\}$	$\{\bar{s}_1^2, \bar{s}_1^2\bar{z}, \bar{s}_1^2\bar{z}^2\}$
$E_6[\theta]$	$\frac{1}{3}(q^3 - 1)$	$\frac{1}{3}(q^3 - 1) + q^3\theta$	$\frac{1}{3}(q^3 - 1) + q^3\theta^2$
$q \equiv 2 \pmod 3$	$\bar{1}$	\bar{s}_1	\bar{s}_1^2
$E_6[\theta]$	$\frac{1}{3}(q^3 + 1)$	$\frac{1}{3}(q^3 + 1) + q^3\theta$	$\frac{1}{3}(q^3 + 1) + q^3\theta^2$

(Here, $g_1 = s_1 u_1 \in \Sigma^F$ is such that g_1 and g_1^{-1} are conjugate in \mathbf{G}^F .)

The resulting values of $E_6[\theta]$ on the conjugacy classes of \mathbf{G}^F that are contained in Σ^F are displayed in Table 4, where z denotes a non-trivial element in $Z(\mathbf{G})$ when $q \equiv 1 \pmod 3$. (Recall that Σ^F splits into 9 classes if $q \equiv 1 \pmod 3$, and into 3 classes if $q \equiv 2 \pmod 3$; these classes are parametrised by representatives of the F -conjugacy classes of $A_{\mathbf{G}}(g_1) \cong Z(\mathbf{H}'_1)$.)

5.6. Twisted type. Let \mathbf{G} be as above (of type E_6) but let now $\tilde{F}: \mathbf{G} \rightarrow \mathbf{G}$ be a Frobenius map (corresponding to an \mathbb{F}_q -rational structure on \mathbf{G}) such that (\mathbf{G}, \tilde{F}) is non-split. Then the induced automorphism $\sigma: \mathbf{W} \rightarrow \mathbf{W}$ is given by conjugation with the longest element $w_0 \in \mathbf{W}$. The permutation $\alpha \mapsto \alpha^\dagger$ of Φ is of order 2, such that $\alpha_1^\dagger = \alpha_6$, $\alpha_3^\dagger = \alpha_5$, $\alpha_2^\dagger = \alpha_4$ and $\alpha_4^\dagger = \alpha_2$. The two cuspidal unipotent character sheaves A_1 and A_2 considered above are also \tilde{F} -stable; see [36, Cor. 20.4] and its proof. In all essential points, we can further argue as above, so we just state the main results. To begin with, the subsystem $\Phi_0 \subseteq \Phi$ is invariant under \dagger . Again, there are three equivalence classes of pairs $(\Phi', w) \in \Xi$ under \sim , where $\Phi' = \Phi_0$; representatives (Φ_0, d_i) , where $d_i \in \mathbf{W}$ has minimal length in $\mathbf{W}(\Phi_0)d_i$ for $i = 1, 2, 3$, are given as follows.

d_i	permutation	σ'_i -classes
$d_1 = 1_{\mathbf{W}}$	$(1, 5)(3, 4)$	9
$d_2 = 431543654$	$(1, 3)(2, 6)(4, 5)$	27
$d'_3 = 423143542314354$	$(1, 6, 5, 3, 2, 4)$	3

Given $t = h(\xi_1, \dots, \xi_6) \in \mathbf{T}_0$, with $\xi \in k^\times$ for all i , we now have

$$\tilde{F}(t) = h(\xi_6^q, \xi_2^q, \xi_5^q, \xi_4^q, \xi_3^q, \xi_1^q) \quad \text{for all } \xi \in k^\times.$$

Recall that $Z(\mathbf{H}') = \{h(\xi, 1, \xi^{-1}, 1, \zeta, \zeta^{-1}) \mid \xi, \zeta \in k^\times, \xi^3 = \zeta^3 = 1\}$; also recall the definition of the \mathbf{G} -conjugacy class \mathcal{C} from §5.1.

Let $t = h(\xi, 1, \xi^{-1}, 1, \zeta, \zeta^{-1}) \in Z(\mathbf{H}')$. Assume first that $q \equiv 1 \pmod 3$. Then $\tilde{F}(t) = t$ if and only if $\xi = \zeta^{-1}$. Thus, there exists some $t \in Z(\mathbf{H}')^{\tilde{F}}$ such that $C_{\mathbf{G}}(t) = \mathbf{H}'$. On the other hand, if $q \equiv 2 \pmod 3$, then one checks that $\tilde{F}(t) = d_2^{-1}td_2$ for all $t \in Z(\mathbf{H}')$. In particular, there exists some $t \in Z(\mathbf{H}')$ such that $C_{\mathbf{G}}(t) = \mathbf{H}'$ and $\tilde{F}(t) = d_2^{-1}td_2$. Hence, in both cases, the class \mathcal{C} is \tilde{F} -stable. We now define Σ as in §5.2; let $g_1 \in \Sigma^{\tilde{F}}$. It follows again that

$$A_{\mathbf{G}}(g_1) \text{ is generated by } \bar{s}_1 \text{ and all } \bar{z} \text{ for } z \in Z(\mathbf{G}).$$

We obtain characteristic functions χ_{g_1, ψ_1} and χ_{g_1, ψ_2} for A_1 and A_2 , respectively, by exactly the same formulae as in §5.3.

Now let us turn to the unipotent characters and almost characters of $\mathbf{G}^{\tilde{F}}$. The unipotent characters are parametrised by the same set $X(\mathbf{W})$ as before in §5.4; the notation for the characters in the 8-element family is now as follows.

$x \in X(\mathbf{W}) :$	(1, 1)	(1, ε)	(g_2 , 1)	(g_3 , 1)	(1, r)	(g_2 , ε)	(g_3 , θ)	(g_3 , θ^2)
$\tilde{\rho}_x \in \text{Unip}(\mathbf{G}^{\tilde{F}}) :$	${}^2E_6[1]$	$[\phi_{12,4}]$	$[\phi_{4,8}]$	$[\phi'_{6,6}]$	$[\phi''_{6,6}]$	$[\phi_{16,5}]$	${}^2E_6[\theta]$	${}^2E_6[\theta^2]$

Here, we use the notation in [4, p. 481]. Thus, $\phi_{12,4}$, $\phi'_{6,6}$ etc. are irreducible characters of $\mathbf{W}^\sigma := \{w \in \mathbf{W} \mid \sigma(w) = w\}$ (a Weyl group of type F_4); then $[\phi_{12,4}]$, $[\phi'_{6,6}]$ etc. are the corresponding irreducible constituents of $\text{Ind}_{\mathbf{B}^{\tilde{F}}}^{\mathbf{G}^{\tilde{F}}}(1)$; characters denoted like ${}^2E_6[1]$ are cuspidal unipotent. (We refer to [4] instead of [33], because the table of unipotent characters for twisted E_6 is not explicitly printed in [33].)

For each $x \in X(\mathbf{W})$ we also have a corresponding unipotent almost character \tilde{R}_x of $\mathbf{G}^{\tilde{F}}$. Since \mathbf{G}, \tilde{F} is not of split type, there is no canonical choice for these almost characters; they are only well-defined up to multiplication by roots of unity. But, by [33, 4.19], we can choose \tilde{R}_x such that, for each $x' \in X(\mathbf{W})$, the multiplicity of $\tilde{\rho}_{x'} \in \text{Unip}(\mathbf{G}^{\tilde{F}})$ in \tilde{R}_x is equal to that of $\rho_{x'} \in \text{Unip}(\mathbf{G}^F)$ in the almost character R_x of \mathbf{G}^F . In particular, this yields the two formulae:

$$\tilde{R}_{(g_3, \theta)} := \frac{1}{3}({}^2E_6[1] + [\phi_{12,4}] - [\phi'_{6,6}] - [\phi''_{6,6}] + 2 \cdot {}^2E_6[\theta] - {}^2E_6[\theta^2]),$$

$$\tilde{R}_{(g_3, \theta^2)} := \frac{1}{3}({}^2E_6[1] + [\phi_{12,4}] - [\phi'_{6,6}] - [\phi''_{6,6}] - {}^2E_6[\theta] + 2 \cdot {}^2E_6[\theta^2]).$$

Furthermore, if $\phi \in \text{Irr}(\mathbf{W})$ is such that x_ϕ belongs to the above 8-element family of $X(\mathbf{W})$, then the above choice leads to the following formula for \tilde{R}_{x_ϕ} :

$$\tilde{R}_{x_\phi} = \tilde{R}_\phi := -\frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \phi(w w_0) R_{T_w}^{\mathbf{G}}(1).$$

(Indeed, according to [33, 4.19], one needs to take the “preferred extension” of ϕ as in [36, 17.2], in order for the formula $\langle \tilde{\rho}_{x'}, \tilde{R}_\phi \rangle = \langle \rho_{x'}, R_\phi \rangle$ to hold; since the a -invariants of all ϕ as above are 3, this leads to the minus sign in the definition of \tilde{R}_ϕ . See also [33, Prop. 7.11].) Now, by the main results of Shoji [54, §4] (see also [54, 4.8, 5.2]), there are scalars $\zeta, \zeta' \in \mathbb{K}$ of absolute value 1 such that

$$\tilde{R}_{(g_3, \theta)} = \zeta \chi_{g_1, \psi_1} \quad \text{and} \quad \tilde{R}_{(g_3, \theta^2)} = \zeta' \chi_{g_1, \psi_2}.$$

By the same argument as in §5.4, we can choose $g_1 \in \Sigma^{\tilde{F}}$ such that g_1 is conjugate to g_1^{-1} in $\mathbf{G}^{\tilde{F}}$. Hence, as before, we already know that $\zeta' = \bar{\zeta} = \zeta = \pm 1$. So it only remains to decide whether ζ equals 1 or -1 .

Proposition 5.7. *Recall that $g_1 \in \Sigma^{\tilde{F}}$ is conjugate to g_1^{-1} in $\mathbf{G}^{\tilde{F}}$. With $\tilde{R}_{(g_3, \theta)}$ and $\tilde{R}_{(g_3, \theta^2)}$ as specified above, we have $\zeta = \zeta' = 1$.*

Proof. As in the proof of Proposition 5.5, we invert the matrix relating unipotent characters and unipotent almost characters of $\mathbf{G}^{\tilde{F}}$; this yields the identity:

$${}^2E_6[\theta] = \frac{1}{3}(\tilde{R}_{80_s} + \tilde{R}_{20_s} - \tilde{R}_{10_s} - \tilde{R}_{90_s} + 2\tilde{R}_{(g_3, \theta)} - \tilde{R}_{(g_3, \theta^2)}).$$

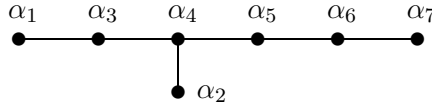
By the above definition of \tilde{R}_ϕ and the formula in Example 3.3, we obtain

$$\tilde{R}_{80_s}(g_1) = \tilde{R}_{20_s}(g_1) = \tilde{R}_{90_s}(g_1) = 0 \quad \text{and} \quad \tilde{R}_{10_s}(g_1) = \varepsilon,$$

where $\varepsilon = \pm 1$ is such that $q \equiv \varepsilon \pmod{3}$. This yields the relation ${}^2E_6[\theta](g_1) = \frac{1}{3}(\zeta q^3 - \varepsilon)$, which implies that $\zeta = 1$ regardless of whether ε is $+1$ or -1 . \square

6. Cuspidal unipotent character sheaves in type E_7

Throughout this section, let \mathbf{G} be simple, simply connected of type E_7 . Let $q = p^f$ (where $f \geq 1$) be such that $F: \mathbf{G} \rightarrow \mathbf{G}$ defines an \mathbb{F}_q -rational structure. Here, \mathbf{G} is of split type; thus, $\sigma = \text{id}_{\mathbf{W}}$ and the permutation $\alpha \mapsto \alpha^\dagger$ of Φ is the identity. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ be the set of simple roots in Φ^+ , where the labelling is chosen as follows.



If $p = 2$, then the cuspidal character sheaves and almost characters have been considered by Hetz [26]. So assume from now on that $p \neq 2$. Let $\alpha_0 \in \Phi$ be the unique root of maximal height and consider the subsystem $\Phi_0 \subseteq \Phi$ of type $A_3 \times A_3 \times A_1$ spanned by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_0\}$. Again, the relevance of this particular example is that Φ_0 occurs in the classification of cuspidal unipotent character sheaves on \mathbf{G} ; see [36, Prop. 20.3] (and also [54, 5.2]). Using CHEVIE, we find that the unique set Δ_0 of simple roots in $\Phi_0 \cap \Phi^+$ is given by

$$\Delta_0 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha'_0 := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \}.$$

Furthermore, there are four equivalence classes of pairs $(\Phi', w) \in \Xi$ under \sim , where $\Phi' = \Phi_0$; representatives (Φ_0, d_i) , where $d_i \in \mathbf{W}$ has minimal length in $\mathbf{W}(\Phi_0)d_i$ for $i = 1, 2, 3, 4$, are given in Table 5, where we use similar notational conventions as in the previous section. (Note that, in the Dynkin diagram of Δ_0 , the node α'_0 is joined to α_3 , not to α_1 .)

Table 5: The subsystem Φ_0 for type E_7

d_i	permutation	σ'_i -classes
$d_1 = 1_{\mathbf{W}}$	$()$	50
$d_2 = 423143542654317654234$	$(1, 6)(3, 5)(4, 7)$	10
$d_3 = 4234542346542347654234$	$(1, 7)(4, 6)$	50
$d_4 = 42314354231435465423143542654$	$(1, 4)(3, 5)(6, 7)$	10

6.1. The subgroup $\mathbf{H}' = \langle \mathbf{T}_0, \mathbf{U}_\alpha (\alpha \in \Phi_0) \rangle$. As in §5.1, every $t \in \mathbf{T}_0$ has a unique expression $t = h(\xi_1, \dots, \xi_7) := \prod_{1 \leq i \leq 7} \alpha_i^\vee(\xi_i)$ where $\xi_i \in k^\times$ for $1 \leq i \leq 7$. By [21, Example 1.5.6], we have

$$Z(\mathbf{G}) = \{h(1, \xi, 1, 1, \xi, 1, \xi) \mid \xi = \pm 1 \in k\} \cong \mathbb{Z}/2\mathbb{Z}.$$

A similar computation shows that

$$Z(\mathbf{H}') = \{h(1, \pm 1, 1, 1, \xi, \xi^2, \xi^{-1}) \mid \xi \in k^\times, \xi^4 = 1\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

Since q is not a power of 2, we have $|Z(\mathbf{G})| = 2$ and $|Z(\mathbf{H}')| = 8$. Given $t = h(1, \pm 1, 1, 1, \xi, \xi^2, \xi^{-1}) \in Z(\mathbf{H}')$ (where $\xi^4 = 1$), we have $C_{\mathbf{G}}(t) = \mathbf{H}'$ if and

only if $\xi^2 \neq 1$. The elements $h(1, -1, 1, 1, 1, 1, 1)$ and $h(1, 1, 1, 1, -1, 1, -1)$ have a centraliser of type $D_6 \times A_1$. Furthermore, one easily checks that

$$d_2^{-1}td_2 = zt, \quad d_3^{-1}td_3 = t^{-1}, \quad d_4^{-1}td_4 = zt^{-1},$$

where $z := h(1, \xi^2, 1, 1, \xi^2, 1, \xi^2) \in Z(\mathbf{G})$. These three relations show that all elements $t \in Z(\mathbf{H}')$ such that $C_{\mathbf{G}}(t) = \mathbf{H}'$ are conjugate in \mathbf{G} . Thus, if \mathcal{C} denotes the \mathbf{G} -conjugacy class of these elements, then

$$F(\mathcal{C}) = \mathcal{C}, \quad \mathcal{C} = \mathcal{C}^{-1} \quad \text{and} \quad \mathcal{C} = Z(\mathbf{G})\mathcal{C}.$$

In order to see that \mathcal{C} is F -stable, we argue as follows. If $q \equiv 1 \pmod 4$, then $F(z) = z$ for all $z \in Z(\mathbf{H}')$. On the other hand, if $q \equiv 3 \pmod 4$, then $F(z) = z^{-1}$ for all $z \in Z(\mathbf{H}')$ and, hence, $Z(\mathbf{H}')^F$ is a Klein four group. If $t \in Z(\mathbf{H}') \cap \mathcal{C}$ as above, then $F(t) = t^{-1} = d_3^{-1}td_3 \in \mathcal{C}$, as required.

As in §5.2, we fix an element $s_1 \in \mathcal{C}^F$ and set $\mathbf{H}'_1 := C_{\mathbf{G}}(s_1)$. We pick a regular unipotent element $u_1 \in \mathbf{H}'_1{}^F$ and let Σ be the \mathbf{G} -conjugacy class of $g_1 := s_1u_1$. Again, we see that Σ is F -stable and $Z(\mathbf{G})\Sigma = \Sigma = \Sigma^{-1}$. Furthermore, $A_{\mathbf{G}}(g_1) \cong Z(\mathbf{H}'_1)$ has order 8 and

$$A_{\mathbf{G}}(g_1) = \langle \bar{s}_1, \bar{z} \rangle \quad \text{where } z \text{ is the non-trivial element of } Z(\mathbf{G}).$$

Now F acts trivially on $A_{\mathbf{G}}(g_1)$, regardless of the congruence class of q modulo 4. Hence, the set Σ^F always splits into 8 conjugacy classes in \mathbf{G}^F (each with centraliser order $8q^7$), which are parametrised by the 8 elements of $A_{\mathbf{G}}(g_1)$. However, now it is less obvious whether we can choose $g_1 \in \Sigma^F$ such that g_1 is conjugate to g_1^{-1} in \mathbf{G}^F . We will come back to this issue in §6.4. (Since $Z(\mathbf{G})$ has order 2, we can not use the argument in Lemma 4.10(b).)

6.2. Cuspidal unipotent character sheaves. By an argument entirely analogous to that in §5.3 (but now using [36, Prop. 20.5] and its proof), we see that there are two F -invariant cuspidal unipotent character sheaves A_1 and A_2 on \mathbf{G} . (Again, they are pulled back from $\tilde{\mathbf{G}} = \mathbf{G}/Z(\mathbf{G})$ via the canonical map $\pi: \mathbf{G} \rightarrow \tilde{\mathbf{G}}$.) The local systems associated with A_1 and A_2 are one-dimensional; they correspond to linear characters ψ_1 and ψ_2 of $A_{\mathbf{G}}(g_1)$ such that

$$\psi_1(\bar{s}_1) = i, \quad \psi_2(\bar{s}_1) = -i, \quad \psi_1(\bar{z}) = \psi_2(\bar{z}) = 1 \quad \text{for } z \in Z(\mathbf{H}'_1)^F,$$

where $i \in \mathbb{K}$ is fixed such that $i^2 = -1$. (Recall that $\bar{g}_1 = \bar{s}_1$.) Thus, ψ_1 and ψ_2 are completely determined, where ψ_2 is the complex conjugate of ψ_1 .

Furthermore, the roots of unity attached to A_1 and A_2 as in §4.2 are $\lambda_{A_1} = i$ and $\lambda_{A_2} = -i$. Using ψ_1 and ψ_2 , we can now write down characteristic functions of A_1 and A_2 , as in §4.2. The values are given as follows, where $1 \neq z \in Z(\mathbf{G})$.

	$\{\bar{1}, \bar{z}\}$	$\{\bar{s}_1^2, \bar{s}_1^2 \bar{z}\}$	$\{\bar{s}_1, \bar{s}_1 \bar{z}\}$	$\{\bar{s}_1^{-1}, \bar{s}_1^{-1} \bar{z}\}$
χ_{g_1, ψ_1}	$q^{7/2}$	$-q^{7/2}$	$iq^{7/2}$	$-iq^{7/2}$
χ_{g_1, ψ_2}	$q^{7/2}$	$-q^{7/2}$	$-iq^{7/2}$	$iq^{7/2}$

Note that, here, we have $\dim \mathbf{G} - \dim \Sigma = \dim \mathbf{T}_0 = 7$. We set $q^{n/2} := \sqrt{q}^n$ for any $n \geq 1$, where $\sqrt{q} \in \mathbb{K}$ has been fixed as described in §4.2.

Table 6: Unipotent almost characters for type E_7

$$\begin{aligned}
 R_{512'_a} = R_{(1,1)} &:= \frac{1}{2}([512'_a] + [512_a] - E_7[\xi] - E_7[-\xi]), \\
 R_{512_a} = R_{(1,\varepsilon)} &:= \frac{1}{2}([512'_a] + [512_a] + E_7[\xi] + E_7[-\xi]), \\
 R_{(g_2,1)} &:= \frac{1}{2}([512'_a] - [512_a] - E_7[\xi] + E_7[-\xi]), \\
 R_{(g_2,\varepsilon)} &:= \frac{1}{2}([512'_a] - [512_a] + E_7[\xi] - E_7[-\xi]).
 \end{aligned}$$

6.3. Unipotent characters and almost characters. Exactly as in §5.4, the unipotent characters of \mathbf{G}^F can be canonically identified with those of $\tilde{\mathbf{G}}^F$, where $\tilde{\mathbf{G}} = \mathbf{G}/Z(\mathbf{G})$. Again, they are parametrised by a certain set $X(\mathbf{W})$ (which only depends on \mathbf{W}). We use the notation in the table on [33, pp. 364–365]; also note the special remarks concerning type E_7 (and E_8) on [33, p. 362]. The unipotent almost characters are also parametrised by $X(\mathbf{W})$. The interesting cases for us are displayed in Table 6, where $\xi = i\sqrt{q} \in \mathbb{K}$. In that table, $512'_a$ and 512_a are irreducible characters of \mathbf{W} ; then $[512'_a]$ and $[512_a]$ are the corresponding constituents of $\text{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}(1)$; the characters $E_7[\pm\xi]$ are cuspidal unipotent. (Note also that the “ g_2 ” in $(g_2, 1)$ and (g_2, ε) has nothing to do with elements in \mathbf{G}^F .)

Now consider the two character sheaves A_1 and A_2 described above, with characteristic functions χ_{g_1, ψ_1} and χ_{g_1, ψ_2} . By the main result of [54, §4] (see also [54, 5.2]), there are scalars $\zeta, \zeta' \in \mathbb{K}$ of absolute value 1 such that

$$R_{(g_2,1)} = \zeta \chi_{g_1, \psi_1} \quad \text{and} \quad R_{(g_2,\varepsilon)} = \zeta' \chi_{g_1, \psi_2}.$$

(Again, in [54, §4], this is proved for $\tilde{\mathbf{G}}$ but the discussion in §6.2 shows that this also holds for \mathbf{G} , with ψ_1 and ψ_2 as above.) By [15, Table 1], the characters $E_7[\xi]$ and $E_7[-\xi]$ are complex conjugate to each other, and their

values lie in the field $\mathbb{Q}(\xi)$. Furthermore, all characters $[\phi]$ (where $\phi \in \text{Irr}(\mathbf{W})$) have their values in $\mathbb{Q}(\sqrt{q})$; see [15, Prop. 5.6]. We conclude that $R_{(g_2,1)}$ and $R_{(g_2,\varepsilon)}$ are complex conjugate to each other, and their values lie in $\mathbb{Q}(i, \sqrt{q})$. Since

$$R_{(g_2,1)}(g_1) = \zeta q^{7/2} \quad \text{and} \quad R_{(g_2,\varepsilon)}(g_1) = \zeta' q^{7/2},$$

we can already conclude that $\zeta' = \bar{\zeta}$.

6.4. On the choice of $g_1 \in \Sigma^F$. We now come back to the issue of finding a “good” representative $g_1 \in \Sigma^F$. Recall that $Z(\mathbf{G})$ has order 2. By Example 4.9, there are precisely two \mathbf{G}^F -conjugacy classes $C, C' \subseteq \Sigma^F$ which have a non-empty intersection with $\mathbf{B}^F \dot{w}_c \mathbf{B}^F$. We let $g_1 \in C \cup C'$. To fix the notation, we let C be the \mathbf{G}^F -conjugacy class associated with Σ (and the choice of \dot{w}_c) as in Corollary 4.8; by the construction in Example 4.9, C' is the \mathbf{G}^F -conjugacy class parametrised by $\bar{z} \in A_{\mathbf{G}}(g_1)$. The table in §6.2 now shows that χ_{g_1, ψ_1} and χ_{g_1, ψ_2} have the same value on all elements in $C \cup C'$. Using the formula in Example 3.3, the (known) character table of \mathbf{W} and the required computations concerning σ' -conjugacy classes in \mathbf{W} , we find that the restrictions of the almost characters $R_{512'_a}$ and R_{512_a} to Σ^F are identically zero. Finally, the relations in Table 6 can be inverted and yield the following relations:

$$\begin{aligned} [512'_a] &= \frac{1}{2}(R_{512'_a} + R_{512_a} + \zeta \chi_{g_1, \psi_1} + \zeta' \chi_{g_1, \psi_2}), \\ [512_a] &= \frac{1}{2}(R_{512'_a} + R_{512_a} - \zeta \chi_{g_1, \psi_1} - \zeta' \chi_{g_1, \psi_2}), \\ E_7[\xi] &= \frac{1}{2}(-R_{512'_a} + R_{512_a} - \zeta \chi_{g_1, \psi_1} + \zeta' \chi_{g_1, \psi_2}), \\ E_7[-\xi] &= \frac{1}{2}(-R_{512'_a} + R_{512_a} + \zeta \chi_{g_1, \psi_1} - \zeta' \chi_{g_1, \psi_2}). \end{aligned}$$

So the above discussion implies that $E_7[\xi](g) = (\bar{\zeta} - \zeta)q^{7/2}$ for all $g \in C \cup C'$. Next recall that $\Sigma = \Sigma^{-1}$. So, by Lemma 4.10(a), we have $\{C^{-1}, C'^{-1}\} = \{C, C'\}$ which implies that $E_7[\xi](g^{-1}) = (\bar{\zeta} - \zeta)q^{7/2}$ for all $g \in C \cup C'$. But the left hand side also equals $E_7[\bar{\xi}](g) = (\zeta - \bar{\zeta})q^{7/2}$. This implies that $\bar{\zeta} = \zeta$ and, consequently, $\zeta = \zeta' = \pm 1$ (since ζ has absolute value 1). Then all the values of $[512_a]$, $[512'_a]$ and $E_7[\pm\xi]$ on Σ^F are determined (up to $\zeta = \pm 1$); see Table 7.

Now, how important is it to determine the scalar $\zeta = \pm 1$ exactly? The above expressions for $E_7[\pm\xi]$ show that $E_7[\xi](g) = E_7[-\xi](g) \in \mathbb{Z}$ for all $g \in \mathbf{G}^F \setminus \Sigma^F$. In other words, the two characters $E_7[\xi]$ and $E_7[-\xi]$ can only (!) be distinguished by their values on elements in Σ^F , where they are given by Table 7. The same is also true for $[512'_a]$ and $[512_a]$. Thus, up to simultaneously exchanging the names of $[512'_a]$, $[512_a]$ and of $E_7[\xi]$, $E_7[-\xi]$, we could

Table 7: Values of $[512'_a]$, $[512_a]$ and $E_7[\pm\xi]$ on Σ^F

$(\zeta = \zeta' = \pm 1)$	$\{\bar{1}, \bar{z}\}$	$\{\bar{s}_1^2, \bar{s}_1^2 \bar{z}\}$	$\{\bar{s}_1, \bar{s}_1 \bar{z}\}$	$\{\bar{s}_1^{-1}, \bar{s}_1^{-1} \bar{z}\}$
$[512'_a]$	$\zeta q^{7/2}$	$-\zeta q^{7/2}$	0	0
$[512_a]$	$-\zeta q^{7/2}$	$\zeta q^{7/2}$	0	0
$E_7[\xi]$	0	0	$-i\zeta q^{7/2}$	$i\zeta q^{7/2}$
$E_7[-\xi]$	0	0	$i\zeta q^{7/2}$	$-i\zeta q^{7/2}$

(The classes labelled by $\bar{1}$ and \bar{z} correspond to C and C' .)

assume “without loss of generality” that $\zeta = 1$. For most applications of character theory, this is entirely sufficient. — However, it is actually possible to determine ζ exactly.

Proposition 6.5. *Recall that $g_1 \in C \cup C'$ where C and C' are the two \mathbf{G}^F -conjugacy classes that are contained in Σ^F and have a non-empty intersection with $\mathbf{B}^F \dot{w}_c \mathbf{B}^F$. Then $\zeta = \zeta' = 1$, that is, $R_{(g_2, 1)} = \chi_{g_1, \psi_1}$ and $R_{(g_2, \varepsilon)} = \chi_{g_1, \psi_2}$.*

Proof. Let us denote the \mathbf{G}^F -conjugacy classes in Table 7 by C_1, \dots, C_8 (from left to right); in particular, $C = C_1$ and $C' = C_2$. We will now try to evaluate the formula in Remark 4.12(b) for elements $g \in \Sigma^F$. For this purpose, we write $X(\mathbf{W}) = X^\circ \cup \{x_1, x_2\}$ where $x_1 = (g_2, 1)$ and $x_2 = (g_2, \varepsilon)$. Since every unipotent character of \mathbf{G}^F is a linear combination of unipotent almost characters, we have

$$[\phi] = [\phi]^\circ + \alpha_1(\phi)R_{x_1} + \alpha_2(\phi)R_{x_2} \quad \text{for each } \phi \in \text{Irr}(\mathbf{W}),$$

where $\alpha_1(\phi), \alpha_2(\phi) \in \mathbb{K}$ and $[\phi]^\circ$ is a linear combination of $\{R_x \mid x \in X^\circ\}$. Setting

$$B := \sum_{\phi \in \text{Irr}(\mathbf{W})} \phi^{(q)}(T_{w_c})\alpha_1(\phi) \quad \text{and} \quad D(g) := \sum_{\phi \in \text{Irr}(\mathbf{W})} \phi^{(q)}(T_{w_c})[\phi]^\circ(g)$$

for $g \in \Sigma^F$, we obtain

$$\sum_{\phi \in \text{Irr}(\mathbf{W})} \phi^{(q)}(T_w)[\phi](g) = D(g) + B \cdot (R_{x_1}(g) + R_{x_2}(g)).$$

Now we note the following. Let $x \in X^\circ$ and consider the possible values of R_x on C_1, \dots, C_8 . Since R_x is a linear combination of unipotent characters, R_x takes the same value on C_{2i-1}, C_{2i} for $i = 1, \dots, 4$ (see Lemma 4.4).

Using Table 7, we see that the values of R_{x_1} on C_1, \dots, C_8 are given by $u, u, -u, -u, iu, iu, -iu, -iu$ and those of R_{x_2} are $u, u, -u, -u, -iu, -iu, iu, iu$ (where $u = \zeta q^{7/2}$). Since R_x is orthogonal to both R_{x_1} and R_{x_2} , we conclude that the values of R_x must be x, x, x, x, y, y, y, y for some $x, y \in \mathbb{K}$. What is important here is that $D(g)$ takes the same value on all $g \in C_1 \cup C_2 \cup C_3 \cup C_4$; let us denote by D_0 this common value of $D(g)$ on $C_1 \cup C_2 \cup C_3 \cup C_4$.

Next, the explicitly known matrix relating unipotent characters and unipotent almost characters shows that

$$\alpha_1(512'_a) = \alpha_2(512'_a) = \frac{1}{2}, \quad \alpha_1(512_a) = \alpha_2(512_a) = -\frac{1}{2},$$

and $\alpha_1(\phi) = \alpha_2(\phi) = 0$ for all $\phi \neq 512'_a, 512_a$. Furthermore, we have

$$(512'_a)^{(q)}(T_{w_c}) = q^{7/2} \quad \text{and} \quad (512_a)^{(q)}(T_{w_c}) = -q^{7/2},$$

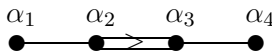
by known results on character values of Hecke algebras (see [23, Example 9.2.9(b)]; these values are readily available within CHEVIE [47]). This yields $B = q^{7/2}$. Furthermore, R_{x_1}, R_{x_2} take value $\zeta q^{7/2}$ on elements in $C_1 \cup C_2$, and value $-\zeta q^{7/2}$ on elements in $C_3 \cup C_4$. Hence, we obtain:

$$\sum_{\phi \in \text{Irr}(\mathbf{W})} \phi^{(q)}(T_w)[\phi](g) = \begin{cases} D_0 + 2\zeta q^7 & \text{if } g \in C_1 \cup C_2, \\ D_0 - 2\zeta q^7 & \text{if } g \in C_3 \cup C_4. \end{cases}$$

By Remark 4.12(b), the left hand side equals $4q^7$ or 0, according to whether $g \in C_1 \cup C_2$ or $g \in \Sigma^F \setminus (C_1 \cup C_2)$. Thus, $0 = D_0 - 2\zeta q^7$ and, consequently, $4q^7 = D_0 + 2\zeta q^7 = 4\zeta q^7$. In particular, $\zeta = 1$. □

7. Cuspidal character sheaves in type F_4

Throughout this section, let \mathbf{G} be simple of type F_4 ; here we have $Z(\mathbf{G}) = \{1\}$. Let $q = p^f$ (where $f \geq 1$) be such that $F: \mathbf{G} \rightarrow \mathbf{G}$ defines an \mathbb{F}_q -rational structure. Here, \mathbf{G} is of split type; thus, $\sigma = \text{id}_{\mathbf{W}}$ and the permutation $\alpha \mapsto \alpha^\dagger$ of Φ is the identity. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the set of simple roots in Φ^+ , where the labelling is chosen as follows.



Except for §7.12 (at the very end), we will assume that $p \neq 2, 3$. The subsystems of Φ occurring in the classification of cuspidal character sheaves are given by Table 8; see [36, Prop. 21.3] and its proof. (We use the same notational conventions as in the previous sections.)

Table 8: Subsystems for type F_4

Φ'	Δ'	d_i	perm.	σ'_i -classes
$A_2 \times A_2$	$\alpha_1, \alpha_3, \alpha_4, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$d_1 = 1\mathbf{w}$	$()$	9
		$d_2 = 232432$	$(1, 4)(2, 3)$	9
$A_3 \times A_1$	$\alpha_1, \alpha_2, \alpha_4, \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$	$d_1 = 1\mathbf{w}$	$()$	10
		$d_2 = 3234323$	$(1, 4)$	10
B_4	$\alpha_1, \alpha_2, \alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4$	$d_1 = 1\mathbf{w}$	$()$	20
$C_3 \times A_1$	$\alpha_2, \alpha_3, \alpha_4, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$d_1 = 1\mathbf{w}$	$()$	20

We begin by working out the center of $\mathbf{H}' = \langle \mathbf{T}_0, \mathbf{U}_\alpha \ (\alpha \in \Phi') \rangle$ in each case.

7.1. The subsystem Φ' of type $A_2 \times A_2$. In this case, we have

$$Z(\mathbf{H}') = \{h(1, 1, \xi, \xi^{-1}) \mid \xi \in k^\times, \xi^3 = 1\} \cong \mathbb{Z}/3\mathbb{Z}.$$

Given $t \in Z(\mathbf{H}')$, we have $C_{\mathbf{G}}(t) = \mathbf{H}'$ if and only if $t \neq 1$. Furthermore, one checks that $\dot{d}_2^{-1}t\dot{d}_2 = t^{-1}$. Hence, the two elements $t \in Z(\mathbf{H}')$ such that $C_{\mathbf{G}}(t) = \mathbf{H}'$ are conjugate in \mathbf{G} . If \mathcal{C} denotes the \mathbf{G} -conjugacy class of these elements, then $\mathcal{C} = \mathcal{C}^{-1}$; furthermore, $F(\mathcal{C}) = \mathcal{C}$. Indeed, if $q \equiv 1 \pmod 3$, then $F(t) = t$ for all $t \in Z(\mathbf{H}')$. On the other hand, if $q \equiv 2 \pmod 3$, then $F(t) = t^{-1}$ for all $t \in Z(\mathbf{H}')$. If $t \in Z(\mathbf{H}') \cap \mathcal{C}$, then $F(t) = t^{-1} = \dot{d}_2^{-1}t\dot{d}_2 \in \mathcal{C}$, as required.

7.2. The subsystem Φ' of type $A_3 \times A_1$. In this case, we have

$$Z(\mathbf{H}') = \{h(1, 1, \xi^2, \xi) \mid \xi \in k^\times, \xi^4 = 1\} \cong \mathbb{Z}/4\mathbb{Z}.$$

Given $t = h(1, 1, \xi^2, \xi) \in Z(\mathbf{H}')$ (where $\xi^4 = 1$), we have $C_{\mathbf{G}}(t) = \mathbf{H}'$ if and only if $\xi^2 \neq 1$. (Note that the element $h(1, 1, 1, -1)$ has a centraliser of type B_4 .) Furthermore, one checks that $\dot{d}_2^{-1}t\dot{d}_2 = h(1, 1, \xi^2, \xi^{-1}) = t^{-1}$. Hence, the two elements $t \in Z(\mathbf{H}')$ such that $C_{\mathbf{G}}(t) = \mathbf{H}'$ are conjugate in \mathbf{G} . If \mathcal{C} denotes the \mathbf{G} -conjugacy class of these elements, then $\mathcal{C} = \mathcal{C}^{-1}$; furthermore, $F(\mathcal{C}) = \mathcal{C}$. Indeed, if $q \equiv 1 \pmod 4$, then $F(t) = t$ for all $t \in Z(\mathbf{H}')$. On the other hand, if $q \equiv 3 \pmod 4$, then $F(t) = t^{-1}$ for all $t \in Z(\mathbf{H}')$. If $t \in Z(\mathbf{H}') \cap \mathcal{C}$, then $F(t) = t^{-1} = \dot{d}_2^{-1}t\dot{d}_2 \in \mathcal{C}$, as required.

7.3. The subsystem Φ' of type B_4 . In this case, we have

$$Z(\mathbf{H}') = \{h(1, 1, \xi, 1) \mid \xi \in k^\times, \xi^2 = 1\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Hence, if $s_1 := h(1, 1, -1, 1) \in Z(\mathbf{H}')$, then $C_{\mathbf{G}}(s_1) = \mathbf{H}'$. (This element s_1 is conjugate to the element $h(1, 1, 1, -1)$ mentioned above.)

7.4. The subsystem Φ' of type $C_3 \times A_1$. In this case, we have

$$Z(\mathbf{H}') = \{h(1, \xi, 1, \xi) \mid \xi \in k^\times, \xi^2 = 1\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Hence, if $s_1 := h(1, -1, 1, -1) \in Z(\mathbf{H}')$, then $C_{\mathbf{G}}(s_1) = \mathbf{H}'$.

7.5. The unipotent class $F_4(a_3)$. In total, there are seven cuspidal character sheaves on \mathbf{G} ; they are all unipotent and F -invariant; see [41, 1.7] and [53, §6, §7]. Let A be any of these seven cuspidal character sheaves and write $A = \text{IC}(\Sigma, \mathcal{E})[\dim \mathcal{E}]$ where Σ is an F -stable conjugacy class of \mathbf{G} and \mathcal{E} is an F -invariant irreducible local system on Σ . In all cases, \mathcal{E} is one-dimensional, so condition (*) in §4.2 is satisfied. Furthermore, let \mathcal{O}_0 be the conjugacy class of the unipotent part of an element in Σ . Then \mathcal{O}_0 is the unipotent class denoted by $F_4(a_3)$ in [57, §5]. (The identification of \mathcal{O}_0 follows from [41, Prop. 1.16] if p is sufficiently large; by Taylor [60], it is enough to assume that $p > 3$. For small values of p , one can also use explicit computations in a matrix realisation of \mathbf{G} and the results of Lawther [29].) We have $A_{\mathbf{G}}(u) \cong \mathfrak{S}_4$ for $u \in \mathcal{O}_0$, and there exists some $u \in \mathcal{O}_0^F$ such that F acts trivially on $A_{\mathbf{G}}(u)$; see Shoji [50]. Thus, \mathcal{O}_0^F splits into five conjugacy classes in \mathbf{G}^F , corresponding to the five conjugacy classes of \mathfrak{S}_4 . As in [50], we denote representatives of those five \mathbf{G}^F -conjugacy classes by x_{14}, \dots, x_{18} . We have $|C_{\mathbf{G}}^\circ(x_i)^F| = q^{12}$ in each case; the groups $A_{\mathbf{G}}(x_i)$ are as follows.

$$\begin{aligned} A_{\mathbf{G}}(x_{14})^F &\cong \mathfrak{S}_4, && \text{(cycle type (1111))}, \\ A_{\mathbf{G}}(x_{15})^F &\cong D_8, && \text{(cycle type (22))}, \\ A_{\mathbf{G}}(x_{16})^F &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, && \text{(cycle type (211))}, \\ A_{\mathbf{G}}(x_{17})^F &\cong \mathbb{Z}/4\mathbb{Z}, && \text{(cycle type (4))}, \\ A_{\mathbf{G}}(x_{18})^F &\cong \mathbb{Z}/3\mathbb{Z}, && \text{(cycle type (31))}. \end{aligned}$$

Thus, the five representatives x_i ($i = 14, \dots, 18$) can be distinguished from each other by the structure of the group $A_{\mathbf{G}}(x_i)^F$. Now let $n \in \mathbb{Z}$ be such that $p \nmid n$. Since \mathcal{O}_0 is uniquely determined by its dimension, each $u \in \mathcal{O}_0$ is conjugate to u^n in \mathbf{G} ; it then also follows that each x_i is conjugate to x_i^n in \mathbf{G}^F , for $i = 14, \dots, 18$. We shall make use of this remark for $n = 2$ in the discussion below.

Now we turn to the detailed description of the seven cuspidal character sheaves of \mathbf{G} , where we follow Lusztig [36, §20, §21] and Shoji [53, §6]. In each case, we will determine the scalar ζ in the identity (\clubsuit'); see §4.3. We deal with the various cases in order of increasing difficulty.

7.6. The cuspidal character sheaves A_3, A_4 . Let $s_1 \in \mathbf{G}^F$ be semisimple such that $\mathbf{H}'_1 = C_{\mathbf{G}}(s_1)$ has a root system Φ' of type $A_2 \times A_2$; recall from §7.1 that $Z(\mathbf{H}'_1) \cong \mathbb{Z}/3\mathbb{Z}$ and this is generated by s_1 . Let $u_1 \in \mathbf{H}'_1{}^F$ be a regular unipotent element and Σ be the conjugacy class of $g_1 := s_1 u_1$. As in §5.2, one sees that $\Sigma = \Sigma^{-1}$ and that $A_{\mathbf{G}}(g_1) \cong \mathbb{Z}/3\mathbb{Z}$ is generated by the image \bar{g}_1 of g_1 in $A_{\mathbf{G}}(g_1)$. By [53, (6.2.4)(c)], there are two cuspidal character sheaves $A_i = \text{IC}(\Sigma, \mathcal{E}_i)[\dim \Sigma]$ where $i = 3, 4$. Let $1 \neq \theta \in \mathbb{K}^\times$ be a fixed third root of unity. Then \mathcal{E}_3 corresponds to the linear character $\psi_3: A_{\mathbf{G}}(g_1) \rightarrow \mathbb{K}^\times$ such that $\psi_3(\bar{g}_1) = \theta$ and \mathcal{E}_4 corresponds to the linear character $\psi_4: A_{\mathbf{G}}(g_1) \rightarrow \mathbb{K}^\times$ such that $\psi_4(\bar{g}_1) = \theta^2$. By [53, (6.4.1)], A_3 is parametrised by $(g_3, \theta) \in X(\mathbf{W})$ and A_4 is parametrised by $(g_3, \theta^2) \in X(\mathbf{W})$. By the main result of [53, §6], there are scalars $\zeta, \zeta' \in \mathbb{K}$ of absolute value 1 such that $R_{(g_3, \theta)} = \zeta \chi_{g_1, \psi_3}$ and $R_{(g_3, \theta^2)} = \zeta' \chi_{g_1, \psi_4}$, where the almost characters $R_{(g_3, \theta)}$ and $R_{(g_3, \theta^2)}$ are defined as the following linear combinations of unipotent characters:

$$\begin{aligned} R_{(g_3, \theta)} &:= \frac{1}{3}([\phi_{12,4}] + F_4^{\text{II}}[1] - [\phi'_{6,6}] - [\phi''_{6,6}] + 2F_4[\theta] - F_4[\theta^2]), \\ R_{(g_3, \theta^2)} &:= \frac{1}{3}([\phi_{12,4}] + F_4^{\text{II}}[1] - [\phi'_{6,6}] - [\phi''_{6,6}] - F_4[\theta] + 2F_4[\theta^2]). \end{aligned}$$

Here, we use the notation in [4, p. 479], with analogous conventions as in the previous sections. Thus, $\phi_{12,4}$ etc. are irreducible characters of \mathbf{W} ; then $[\phi_{12,4}]$ etc. are the corresponding irreducible constituents of $\text{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}(1)$; characters denoted like $F_4^{\text{II}}[1]$ are cuspidal unipotent. (In this section we refer to [4] instead of [33], because the full 21×21 Fourier matrix related to type F_4 is printed on [4, p. 456], and that matrix will be needed for several arguments below.) By Lemma 4.10(b), we can choose $g_1 \in \Sigma^F$ to be conjugate to g_1^{-1} in \mathbf{G}^F . By an argument analogous to that in §5.4, one sees that $R_{(g_3, \theta)}$ and $R_{(g_3, \theta^2)}$ are complex conjugate to each other. So we conclude that

$$\zeta = \zeta' = \pm 1 \quad \text{and} \quad R_{(g_3, \theta)}(g_1) = R_{(g_3, \theta^2)}(g_1) = \zeta q^2.$$

Inverting the matrix relating unipotent characters and unipotent almost characters, we obtain the following relation:

$$F_4[\theta] = \frac{1}{3}(R_{(12,4)} + R_{(1, \lambda^3)} - R_{(6,6)'} - R_{(6,6)''} + 2R_{(g_3, \theta)} - R_{(g_3, \theta^2)}),$$

where we just write, for example, $R_{(12,4)}$ instead of $R_{\phi_{(12,4)}}$. Using CHEVIE and the formula in Example 3.3, we find that

$$R_{(12,4)}(g_1) = R_{(6,6)''}(g_1) = 0, \quad R_{(6,6)'}(g_1) = 1.$$

By [53, (6.2.2)], the pair $(1, \lambda^3)$ also parametrises a cuspidal character sheaf, which will be supported on a conjugacy class distinct from Σ . By the main result of [53, §6], a characteristic function of that character sheaf equals $R_{(1, \lambda^3)}$, up to multiplication by a scalar. Hence, we have $R_{(1, \lambda^3)}(g_1) = 0$ and we obtain $F_4[\theta](g_1) = \frac{1}{3}(-1 + \zeta q^2)$. Since the left hand side is an algebraic integer, this forces that $\zeta = 1$. Thus, we have shown that

$$(a) \quad R_{(g_3, \theta)} = \chi_{g_1, \psi_3} \quad \text{and} \quad R_{(g_3, \theta^2)} = \chi_{g_1, \psi_4};$$

recall that, here, we fixed $g_1 \in \Sigma^F$ such that g_1 is conjugate to g_1^{-1} in \mathbf{G}^F . The values of $F_4[\theta]$ and $F_4[\theta^2]$ on Σ^F are given by the following table.

	$\bar{1}$	\bar{s}_1	\bar{s}_1^2
$F_4[\theta]$	$\frac{1}{3}(q^2 - 1)$	$\frac{1}{3}(q^2 - 1) + q^2\theta$	$\frac{1}{3}(q^2 - 1) + q^2\theta^2$
$F_4[\theta^2]$	$\frac{1}{3}(q^2 - 1)$	$\frac{1}{3}(q^2 - 1) + q^2\theta^2$	$\frac{1}{3}(q^2 - 1) + q^2\theta$

This table also shows that $g_1 \in \Sigma^F$ is uniquely determined (up to \mathbf{G}^F -conjugation) by the property that g_1 is conjugate to g_1^{-1} in \mathbf{G}^F .

7.7. The cuspidal character sheaves A_5, A_6 . Let $s_1 \in \mathbf{G}^F$ be semisimple such that $\mathbf{H}'_1 = C_{\mathbf{G}}(s_1)$ has a root system Φ' of type $A_3 \times A_1$; recall from §7.2 that $Z(\mathbf{H}'_1) \cong \mathbb{Z}/4\mathbb{Z}$ and this is generated by s_1 . Let $u_1 \in \mathbf{H}'_1{}^F$ be a regular unipotent element and Σ be the conjugacy class of $g_1 := s_1 u_1$. As above, one sees that $\Sigma = \Sigma^{-1}$ and that $A_{\mathbf{G}}(g_1) \cong \mathbb{Z}/4\mathbb{Z}$ is generated by $\bar{g}_1 \in A_{\mathbf{G}}(g_1)$. By [53, (6.2.4)(d)], there are two cuspidal character sheaves $A_i = \text{IC}(\Sigma, \mathcal{E}_i)[\dim \Sigma]$ where $i = 5, 6$; here, \mathcal{E}_5 corresponds to the linear character $\psi_5: A_{\mathbf{G}}(g_1) \rightarrow \mathbb{K}^\times$ such that $\psi_5(\bar{g}_1) = i$ (where $i^2 = -1$ in \mathbb{K}) and \mathcal{E}_6 corresponds to the linear character $\psi_6: A_{\mathbf{G}}(g_1) \rightarrow \mathbb{K}^\times$ such that $\psi_6(\bar{g}_1) = -i$. By [53, (6.4.1)], A_5 is parametrised by $(g_4, i) \in X(\mathbf{W})$ and A_6 is parametrised by $(g_4, -i) \in X(\mathbf{W})$. By the main result of [53, §6], there are scalars $\zeta, \zeta' \in \mathbb{K}$ of absolute value 1 such that $R_{(g_4, i)} = \zeta \chi_{g_1, \psi_5}$ and $R_{(g_4, -i)} = \zeta' \chi_{g_1, \psi_6}$, where

$$R_{(g_4, i)} := \frac{1}{4}([\phi_{12,4}] - [\phi'_{9,6}] + [\phi'_{1,12}] - F_4^{\text{II}}[1] - [\phi''_{9,6}] - F_4^{\text{I}}[1] + [\phi''_{1,12}] + [\phi_{4,8}] + 2F_4[i] - 2F_4[-i]);$$

there is a similar expression for $R_{(g_4, -i)}$ where the roles of $F_4[i]$ and $F_4[-i]$ are interchanged. By Lemma 4.10(b), we can choose $g_1 \in \Sigma^F$ to be conjugate in \mathbf{G}^F to g_1^{-1} . As in §7.6, we conclude that $\zeta = \zeta' = \pm 1$. Inverting the matrix relating unipotent characters and unipotent almost characters, we obtain:

$$F_4[i] = \frac{1}{4}(R_{(12,4)} - R_{(9,6)'} + R_{(1,12)'} - R_{(1, \lambda^3)} - R_{(9,6)''})$$

$$- R_{(g'_2, \varepsilon)} + R_{(1,12)''} + R_{(4,8)} + 2R_{(g_4,i)} - 2R_{(g_4,-i)}.$$

Using Example 3.3, we find that $R_\phi(g) = 0$ for all $g \in \Sigma^F$ and all $\phi \in \text{Irr}(\mathbf{W})$ occurring in the sum on the right hand side. Again, by [53, (6.2.2)], the pair (g'_2, ε) also parametrises a cuspidal character sheaf, which will be supported on a conjugacy class distinct from Σ . By the main result of [53, §6], a characteristic function of that character sheaf equals $R_{(g'_2, \varepsilon)}$, up to multiplication by a scalar. Hence, we also have $R_{(g'_2, \varepsilon)}(g) = 0$ for all $g \in \Sigma^F$. A similar argument shows that $R_{(1, \lambda^3)}(g_1) = 0$. This yields the following table for the values of $F_4[i]$ on Σ^F :

	$\bar{1}$	\bar{g}_1^2	\bar{g}_1	\bar{g}_1^{-1}
$F_4[i]$	0	0	ζiq^2	$-\zeta iq^2$
$F_4[-i]$	0	0	$-\zeta iq^2$	ζiq^2

We can now draw the following conclusions. Let C_1, C_2, C_3, C_4 be the four conjugacy classes of \mathbf{G}^F into which Σ^F splits (not necessarily ordered as in the above table). We have $\Sigma = \Sigma^{-1}$, so taking inverses permutes the four classes. The table shows that we can arrange the notation such that $C_4 = C_3^{-1}$, where $C_3 = \text{Sh}_{\mathbf{G}}(C_1)$ and $C_4 = \text{Sh}_{\mathbf{G}}(C_2)$; see §4.1(a). Since $g_1 \in \Sigma^F$ is conjugate in \mathbf{G}^F to g_1^{-1} , this forces that $C_1 = C_1^{-1}$, $C_2 = C_2^{-1}$ and $g_1 \in C_1 \cup C_2$. By Lemma 4.10(b), we can further fix the notation such that $C_1 \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F \neq \emptyset$ and $C_2 \cap \mathbf{B}^F \dot{w}_c \mathbf{B}^F = \emptyset$. Then we claim:

$$(a) \quad \zeta = \begin{cases} 1 & \text{if } g_1 \in C_1, \\ -1 & \text{if } g_1 \in C_2. \end{cases}$$

This is seen by an argument entirely analogous to the proof of Proposition 6.5, based on the formula in Remark 4.12(b). The data required for that argument (that is, the constants $\alpha_1(\phi)$, $\alpha_2(\phi)$ and the values $\phi^{(q)}(T_{w_c})$) are now given as follows. We have $\alpha_1(\phi) = \alpha_2(\phi) = \frac{1}{4}$ for $\phi \in \{\phi'_{1,12}, \phi''_{1,12}, \phi_{4,8}, \phi_{12,4}\}$, and $\alpha_1(\phi) = \alpha_2(\phi) = 0$ otherwise; furthermore, if $\alpha_1(\phi) \neq 0$, then $\phi^{(q)}(T_{w_c}) = q^2$; we omit further details.

7.8. The cuspidal character sheaf A_1 . Let Σ be the unipotent class of \mathbf{G} denoted by $F_4(a_3)$, as already introduced in §7.5. We take $g_1 := x_{14} \in \Sigma^F$; hence, F acts trivially on $A_{\mathbf{G}}(g_1) \cong \mathfrak{S}_4$. We also remarked in §7.5 that g_1 is conjugate in \mathbf{G}^F to g_1^{-1} . As in [53, (6.2.4)(a)], there is a cuspidal character sheaf $A_1 = \text{IC}(\Sigma, \mathcal{E})[\dim \Sigma]$ where \mathcal{E} corresponds to the sign character $\text{sgn} \in \text{Irr}(A_{\mathbf{G}}(g_1))$. In [53, §6], it is not stated explicitly to which parameter in $X(\mathbf{W})$ the character sheaf A_1 corresponds, but this is easily found as follows, using

the information already available from §7.6. (See also the argument in [38, Lemma 8.8].) We claim that A_1 is parametrised by $(1, \lambda^3) \in X(\mathbf{W})$. Assume, if possible, that this is not the case. By Shoji's results [51] on the Green functions of \mathbf{G}^F , we can compute $R_\phi(g_1)$ for any $\phi \in \text{Irr}(\mathbf{W})$. (These results are known to hold whenever $p \neq 2, 3$; see [54, Theorem 5.5] and [14, §3].) In particular, we obtain $R_{(6,6)''}(g_1) = 0$, $R_{(6,6)'}(g_1) = 2q^4$ and $R_{(12,4)}(g_1) = q^4$. Since $R_{(g_3, \theta)}$ and $R_{(g_3, \theta^2)}$ are zero on unipotent elements (see §7.6), we conclude that $F_4[\theta](g_1) = \frac{1}{3}(q^4 - 2q^4) = -q^4/3$, contradiction since $p \neq 3$. Hence, A_1 is parametrised by $(1, \lambda^3)$. By the main result of [53, §6], there is a scalar $\zeta \in \mathbb{K}$ of absolute value 1 such that $R_{(1, \lambda^3)} = \zeta \chi_{g_1, \text{sgn}}$. The exact expression of $R_{(1, \lambda^3)}$ as a linear combination of 21 unipotent characters is obtained from the Fourier matrix on [4, p. 456] and the list of labels for unipotent characters on [4, p. 479]; we will not print it here.

It was first shown by Kawanaka [28, §4] that $\zeta = 1$, assuming that p, q are sufficiently large; Lusztig [38, 8.6, 8.12] shows this assuming that q satisfies a certain congruence condition. Since Kawanaka's results on generalised Gelfand–Graev representations are now known to hold whenever q is a power of a good prime p (see Taylor [60]), we can conclude that $\zeta = 1$ holds unconditionally (but recall our standing assumption that $p > 3$).

We can also argue as follows. Consider again the formula for $F_4[i]$ in §7.7. Using Shoji's results on Green functions, we can compute the values of R_ϕ on unipotent elements, for all ϕ occurring in that formula. Furthermore, we have $R_{(g_2, \varepsilon)}(g_1) = R_{(g_4, \pm i)}(g_1) = 0$. This yields that $F_4[i](g_1) = -\frac{1}{4}q^4(\zeta q^2 - 1)$. Since g_1 is \mathbf{G}^F -conjugate to g_1^{-1} , we have $\zeta = \bar{\zeta}$. Since $F_4[i](g_1)$ is an algebraic integer, we must have $\zeta = 1$.

Table 9: Some character values on the unipotent class $F_4(a_3)$

	x_{14} (1111)	x_{15} (22)	x_{16} (211)	x_{17} (4)	x_{18} (31)
$[\phi'_{1,12}]$	$\frac{1}{8}q^4(q^2-1)+3q^4$	$\frac{1}{8}q^4(q^2-1)$	$\frac{1}{8}q^4(1-q^2)$	$\frac{1}{8}q^4(1-q^2)$	$\frac{1}{8}q^4(q^2-1)$
$[\phi''_{1,12}]$	$\frac{1}{8}q^4(q^2-1)$	$\frac{1}{8}q^4(q^2-1)+q^4$	$\frac{1}{8}q^4(1-q^2)$	$\frac{1}{8}q^4(1-q^2)$	$\frac{1}{8}q^4(q^2-1)$
$F_4[-1]$	$\frac{1}{4}q^4(1-q^2)$	$\frac{1}{4}q^4(1-q^2)$	$\frac{1}{4}q^4(q^2-1)+q^4$	$\frac{1}{4}q^4(q^2-1)$	$\frac{1}{4}q^4(1-q^2)$
$F_4[i]$	$\frac{1}{4}q^4(1-q^2)$	$\frac{1}{4}q^4(1-q^2)$	$\frac{1}{4}q^4(q^2-1)$	$\frac{1}{4}q^4(q^2-1)+q^4$	$\frac{1}{4}q^4(1-q^2)$

7.9. Character values on $F_4(a_3)$. Once $R_{(1, \lambda^3)}$ has been determined, we can determine all character values on $\mathbf{G}_{\text{uni}}^F$, where \mathbf{G}_{uni} denotes the unipotent variety of \mathbf{G} . Indeed, the 25 unipotent almost characters R_ϕ (for $\phi \in \text{Irr}(\mathbf{W})$)

remain linearly independent upon restriction to $\mathbf{G}_{\text{uni}}^F$; they are explicitly computed by Shoji [51]. (As mentioned above, Shoji’s results remain valid whenever $p \neq 2, 3$.) Hence, together with the “cuspidal” almost character $R_{(1, \lambda^3)}$, we obtain 26 linearly independent functions on $\mathbf{G}_{\text{uni}}^F$. Since there are also 26 unipotent conjugacy classes of \mathbf{G}^F (see [50, Theorem 2.1]), we obtain a basis for the space of class functions on $\mathbf{G}_{\text{uni}}^F$. Note that all the remaining unipotent almost characters are orthogonal to the functions in that basis, which implies that they are identically zero on $\mathbf{G}_{\text{uni}}^F$. In §7.11 below, we shall need the values of some unipotent characters on \mathcal{O}_0^F , with \mathcal{O}_0 as in §7.5. The values displayed in Table 9 will allow us to distinguish the various \mathbf{G}^F -conjugacy classes contained in \mathcal{O}_0^F . These values are easily obtained from the functions `UnipotentCharacters` and `ICCTable` in Michel’s version of CHEVIE [47].

7.10. The cuspidal character sheaf A_7 . Let $s_1 := h(1, 1, -1, 1) \in \mathbf{T}_0^F$ and $\mathbf{H}'_1 := C_{\mathbf{G}}(s_1)$; then \mathbf{H}'_1 has a root system Φ' of type B_4 (see §7.3); recall that $Z(\mathbf{H}'_1) \cong \mathbb{Z}/2\mathbb{Z}$ and this is generated by s_1 . Consider the natural isogeny $\beta: \mathbf{H}'_1 \rightarrow \overline{\mathbf{H}}'_1 := \text{SO}_9(k)$ (defined over \mathbb{F}_q). Let \mathcal{O} be the unipotent class of \mathbf{H}'_1 such that the elements $\beta(u) \in \overline{\mathbf{H}}'_1$, for $u \in \mathcal{O}$, have Jordan type $(5, 3, 1)$. Let Σ be the conjugacy class of $s_1 u$, where $u \in \mathcal{O}$. Now \mathcal{O} is F -stable and so Σ is also F -stable. By Shoji [50, Table 4], and the correction discussed by Fleischmann–Janiszczak [13, p. 233], we have:

- (a) There exists an element $g_1 \in \Sigma^F$ such that $A_{\mathbf{G}}(g_1)$ is dihedral of order 8 and F acts trivially on $A_{\mathbf{G}}(g_1)$; we have $|C_{\mathbf{G}}(g_1)^F| = 8q^8$.

Thus, the set Σ^F splits into five conjugacy classes in \mathbf{G}^F , with centraliser orders $8q^8, 8q^8, 4q^4, 4q^4, 4q^4$. So there are two possibilities for the \mathbf{G}^F -conjugacy class of g_1 as in (a). (We just choose one of them; this choice does not affect the result at the end. By §4.1(a) and [9, Chap. I, Prop. 7.2], we also see that the two classes are interchanged by the Shintani map $\text{Sh}_{\mathbf{G}}$.) Now, by [53, (6.2.4)(e)], there is a cuspidal character sheaf $A_7 = \text{IC}(\Sigma, \mathcal{E})[\dim \Sigma]$ where \mathcal{E} corresponds to the sign character $\text{sgn} \in \text{Irr}(A_{\mathbf{G}}(g_1))$. By [53, (6.2.2)], A_7 is parametrised either by the pair (g_2, ε) or by the pair (g'_2, ε) in $X(\mathbf{W})$. We can easily fix this as follows. We note that the eigenvalue $\lambda_{A_7} = \text{sgn}(\bar{g}_1)$ in §4.2 must be 1 since \bar{g}_1 is in the center of $A_{\mathbf{G}}(g_1)$. But there are also certain eigenvalues for the almost characters, where $\lambda_{R_x} = -1$ for $x = (g_2, \varepsilon)$, and $\lambda_{R_x} = 1$ for $x = (g'_2, \varepsilon)$; see [53, (6.2.2)]. By the main result of [53, §6], there is a scalar $\zeta \in \mathbb{K}$ of absolute value 1 such that $R_x = \zeta \chi_{(g_1, \text{sgn})}$ where $x \in \{(g_2, \varepsilon), (g'_2, \varepsilon)\}$ and where the eigenvalues of the character sheaves do match those of the almost characters (see [54, 4.6]). Hence,

- (b) A_7 is parametrised by $(g'_2, \varepsilon) \in X(\mathbf{W})$ and $R_{(g'_2, \varepsilon)} = \zeta \chi_{g_1, \text{sgn}}$.

The exact expression of $R_{(g'_2, \varepsilon)}$ as a linear combination of 18 unipotent characters is obtained from the Fourier matrix on [4, p. 456] and the list of labels for unipotent characters on [4, p. 479]; we will not print it here. We claim:

(c) With g_1 as in (a), we have $\zeta = 1$.

This is seen as follows. We use again the following identity from §7.7:

$$F_4[i] = \frac{1}{4}(R_{(12,4)} - R_{(9,6)'} + R_{(1,12)'} - R_{(1,\lambda^3)} - R_{(9,6)''} - R_{(g'_2, \varepsilon)} + R_{(1,12)''} + R_{(4,8)} + 2R_{(g_4,i)} - 2R_{(g_4,-i)}).$$

Now all R_ϕ ($\phi \in \text{Irr}(\mathbf{W})$) are rational-valued. Since $\overline{F_4[i]} = F_4[-i]$ and $\overline{R_{(g_4,i)}} = R_{(g_4,-i)}$, we conclude that $R_{(g'_2, \varepsilon)}$ is invariant under complex conjugation and, hence, $\zeta = \pm 1$. Now evaluate $F_4[i]$ on $g_1 \in \Sigma^F$. Note that $R_{(1,\lambda^3)}$ and $R_{(g_4, \pm i)}$ have support on conjugacy classes that are distinct from Σ^F and, hence, their value is zero on g_1 ; see §7.7 and §7.8. By Example 3.5, we obtain

$$\begin{aligned} R_{(9,6)''}(g_1) &= R_{(4,8)}(g_1) = R_{(1,12)''}(g_1) = q^2, \\ R_{(12,4)}(g_1) &= R_{(9,6)'}(g_1) = R_{(1,12)'}(g_1) = 0. \end{aligned}$$

(Recall that g_1 is chosen such that $|C_{\mathbf{G}}(g_1)^F| = 8q^8$.) Since $R_{(g'_2, \varepsilon)}(g_1) = \zeta q^4$, we obtain $F_4[i](g_1) = \frac{1}{4}q^2(1 - \zeta q^2)$. Since the left hand side is an algebraic integer, we deduce that $\zeta = 1$. Thus, (c) is proved. Finally, we note:

(d) If $g_1 = s_1 u_1$ is as in (a), then u_1 is \mathbf{G}^F -conjugate to x_{14} or x_{15} .

Indeed, since $C_{\mathbf{G}}(g_1) \subseteq C_{\mathbf{G}}(g_1^2) = C_{\mathbf{G}}(u_1^2)$ and u_1^2 is \mathbf{G}^F -conjugate to $u_1 \in \mathcal{O}_0$, we conclude that 8 divides $|C_{\mathbf{G}}(u_1)^F|$. Hence, the only possibilities are that u_1 is \mathbf{G}^F -conjugate to x_{14} or x_{15} . I conjecture that for one of the two possibilities of $g_1 = s_1 u_1$ as in (a), we do have that u_1 is \mathbf{G}^F -conjugate to x_{14} (but the choice of that g_1 may depend on $q \pmod 4$).

7.11. The cuspidal character sheaf A_2 . Let $s_1 \in \mathbf{G}^F$ be semisimple such that $\mathbf{H}'_1 = C_{\mathbf{G}}(s_1)$ has a root system Φ' of type $C_3 \times A_1$; recall from §7.4 that $Z(\mathbf{H}'_1) \cong \mathbb{Z}/2\mathbb{Z}$ and this is generated by s_1 . Now we have a natural isogeny $\beta: \text{Sp}_4(k) \times \text{SL}_2(k) \rightarrow \mathbf{H}'_1$ (defined over \mathbb{F}_q). Let \mathcal{O} be the unipotent class of \mathbf{H}'_1 that corresponds to unipotent elements of Jordan type $(4, 2) \times (2)$ under β . We start by picking any element $u_1 \in \mathcal{O}^F$ and let Σ be the conjugacy class of $g_1 := s_1 u_1$. We have $\dim \mathbf{G} - \dim \Sigma = 6$ and $|C_{\mathbf{G}}(g_1)^F| = 4q^6$; one easily sees that $\Sigma = \Sigma^{-1}$. Now there is some $1 \neq a \in A_{\mathbf{G}}(g_1)$ such that

$$A_{\mathbf{G}}(g_1) = \langle \bar{g}_1 \rangle \times \langle a \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \quad (\text{with trivial } F\text{-action}).$$

By [53, (6.2.4)(b)], there is a cuspidal character sheaf $A_2 = \text{IC}(\Sigma, \mathcal{E})[\dim \Sigma]$ where \mathcal{E} corresponds to a non-trivial $\psi \in \text{Irr}(A_{\mathbf{G}}(g_1))$ (further specified below). By [53, (6.2.2)], A_2 is parametrised either by the pair (g_2, ε) or by the pair (g'_2, ε) in $X(\mathbf{W})$. By §7.10(b), we conclude that A_2 must be parametrised by (g_2, ε) ; in particular, $\psi(\bar{g}_1) = \lambda_{A_2} = -1$. We can now also fix the element $a \in A_{\mathbf{G}}(g_1)$ such that $\psi(a) = 1$. By the main result of [53, §6], there is a scalar $\zeta \in \mathbb{K}$ of absolute value 1 such that $R_{(g_2, \varepsilon)} = \zeta \chi_{g_1, \psi}$, where

$$R_{(g_2, \varepsilon)} := \frac{1}{4}([\phi_{12,4}] + [\phi'_{9,6}] - [\phi'_{1,12}] - F_4^{\mathbb{I}}[1] - 2[\phi_{16,5}] + 2F_4[-1] + [\phi''_{9,6}] - F_4^{\mathbb{I}}[1] - [\phi''_{1,12}] + [\phi_{4,8}]).$$

Let C_1, C_2, C_3, C_4 be the four \mathbf{G}^F -conjugacy classes into which Σ^F splits (initially ordered in no particular way). For each i , we denote $C_i^{[2]} := \{g^2 \mid g \in C_i\}$. Writing $g_1 = s_1 u_1$ as above, we have $u_1 \in \mathcal{O}_0$; furthermore, u_1, u_1^2 are \mathbf{G}^F -conjugate and so $g_1^2 = u_1^2 \in \mathcal{O}_0$ (see §7.5). We claim that the notation can be arranged such that

(a)
$$x_{14} \in C_1^{[2]}, \quad x_{15} \in C_2^{[2]} \quad \text{and} \quad x_{16} \in C_3^{[2]} = C_4^{[2]}.$$

Depending on how we choose $g_1 \in \Sigma^F$, the scalar ζ is then determined as follows.

(b)
$$\zeta = \begin{cases} 1 & \text{if } g_1 \in C_1 \cup C_2, \\ -1 & \text{if } g_1 \in C_3 \cup C_4. \end{cases}$$

This is proved as follows. Inverting the matrix relating unipotent characters and unipotent almost characters, we obtain:

$$F_4[-1] = \frac{1}{4}(R_{(12,4)} + R_{(9,6)'} - R_{(1,12)'} - R_{(1,\lambda^3)} - 2R_{(16,5)} + 2R_{(g_2, \varepsilon)} + R_{(9,6)''} - R_{(g'_2, \varepsilon)} - R_{(1,12)''} + R_{(4,8)}).$$

Now we evaluate this on $g_1 \in \Sigma^F$. By §7.8 and §7.10, we have $R_{(1,\lambda^3)}(g_1) = R_{(g'_2, \varepsilon)}(g_1) = 0$. By a computation entirely analogous to that in Example 3.5, we obtain $R_{(16,5)}(g_1) = q$ and

$$R_{(12,4)}(g_1) = R_{(9,6)'}(g_1) = R_{(9,6)''}(g_1) = R_{(1,12)'}(g_1) = R_{(1,12)''}(g_1) = 0;$$

this does not depend on how we choose $g_1 \in \Sigma^F$. Since $R_{(g_2, \varepsilon)}$ takes the values $\zeta q^3, \zeta q^3, -\zeta q^3, -\zeta q^3$ on the representatives in Σ^F parametrised by $\bar{1}, a, \bar{g}_1, a\bar{g}_1$, this yields the following values for $F_4[-1]$ on Σ^F .

	$\bar{1}$	a	\bar{g}_1	$a\bar{g}_1$
$F_4[-1]$	$\frac{1}{2}q(\zeta q^2 - 1)$	$\frac{1}{2}q(\zeta q^2 - 1)$	$-\frac{1}{2}q(\zeta q^2 + 1)$	$-\frac{1}{2}q(\zeta q^2 + 1)$

By [15, Table 1], the character $F_4[-1]$ is rational-valued, so we must have $\zeta = \pm 1$. Regardless of whether ζ equals 1 or -1 , two of the above values are $\frac{1}{2}q(q^2 - 1)$, and two of them are $-\frac{1}{2}q(q^2 + 1)$. Thus, two of the above values are even integers, and two of them are odd integers. Now compare with Table 9:

$$F_4[-1](x_{16}) \equiv 1 \pmod{2} \quad \text{and} \quad F_4[-1](x_i) \equiv 0 \pmod{2} \quad \text{for } i \neq 16.$$

By a well-known fact from the general character theory of finite groups, we have $F_4[-1](g_1^2) \equiv F_4[-1](g_1) \pmod{2}$. Hence, if $g_1 \in \Sigma^F$ is such that $F_4[-1](g_1)$ is odd, then g_1^2 must be \mathbf{G}^F -conjugate to x_{16} . Since there are two \mathbf{G}^F -conjugacy classes in Σ^F on which the value of $F_4[-1]$ is odd, we conclude that $x_{16} \in C_i^{[2]}$ for two values of $i \in \{1, 2, 3, 4\}$; we arrange the notation such that these two values are $i = 3$ and $i = 4$. Now choose $g_1 \in \Sigma^F$ such that $g_1 \in C_3 \cup C_4$. Since $F_4[-1](g_1)$ is given by the entry corresponding to $\bar{1} \in A_{\mathbf{G}}(g_1)$ in the above table, we conclude that $\frac{1}{2}q(\zeta q^2 - 1)$ must be odd and so $\zeta = -1$. Thus, (a) and (b) are proved as far as C_3 and C_4 are concerned. On the other hand, let us choose $g_1 \in \Sigma^F \setminus (C_3 \cup C_4)$. Then $F_4[-1](g_1) = \frac{1}{2}q(\zeta q^2 - 1)$ must be even and so $\zeta = 1$. So all that remains to be done is to identify $i, j \in \{14, \dots, 18\}$ such that $x_i \in C_1^{[2]}$ and $x_j \in C_2^{[2]}$. For this purpose, we consider the characters $[\phi'_{1,12}]$ and $[\phi''_{1,12}]$.

Using the ingredients of the CHEVIE function `LusztigMapb` explained in [47, §7] (which relies on the theoretical fact that the indicator function of a \mathbf{G}^F -conjugacy class is “uniform”, see [17, §8]), we can compute $\sum_{g \in \Sigma^F} \rho(g)$ for any $\rho \in \text{Unip}(\mathbf{G}^F)$. Since all elements in Σ^F have the same centraliser order, we can actually compute the sum of the four values of ρ on C_1, C_2, C_3, C_4 . Applying this to $\rho = [\phi'_{1,12}]$, we find that the result is $-q$. Consequently, the four values of $[\phi'_{1,12}]$ on C_1, C_2, C_3, C_4 cannot all have the same parity. Hence, there exists some $g \in \Sigma^F$ such that $[\phi'_{1,12}](g) \equiv [\phi'_{1,12}](x_{14}) \pmod{2}$. But then we also have

$$[\phi'_{1,12}](g^2) \equiv [\phi'_{1,12}](g) \equiv [\phi'_{1,12}](x_{14}) \pmod{2}.$$

Since $[\phi'_{1,12}](x_i) \not\equiv [\phi'_{1,12}](x_{14}) \pmod{2}$ for $i \neq 14$ (see Table 9), we conclude that g^2 is \mathbf{G}^F -conjugate to x_{14} . Thus, we can arrange the notation such that $x_{14} \in C_1^{[2]}$. Then a completely analogous argument using the character $[\phi''_{1,12}]$ shows that $x_{15} \in C_2^{[2]}$. Thus, (a) and (b) are proved. The above table of values also shows that the values of $F_4[-1]$ on the classes parametrised by $\bar{1}$ and \bar{g}_1

have a different parity; similarly for a and $a\bar{g}_1$. Hence, we can fix the notation for C_3 and C_4 such that $C_3 = \text{Sh}_{\mathbf{G}}(C_1)$ and $C_4 = \text{Sh}_{\mathbf{G}}(C_2)$ (see §4.1(a)).

Finally, we remark that we can also obtain an explicit representative in Σ^F . Indeed, using CHEVIE, we can easily compute the full \mathbf{W} -orbit of s_1 ; by inspection, $s'_1 := h(-1, -1, 1, 1) \in \mathbf{T}_0^F$ belongs to that orbit, that is, s'_1 is conjugate to s_1 in \mathbf{G}^F . Using the explicit expression (in terms of Chevalley generators of \mathbf{G}^F) for x_{16} in [50, Table 5], we can check that s'_1 commutes with x_{16} . Hence, we have $g_1 := s'_1 x_{16} \in \Sigma^F$; since $g_1 = x_{16}^2$, we must have $\zeta = -1$ for this choice of g_1 .

7.12. The cases where $p = 2, 3$. In the above discussion, we assumed that $p \neq 2, 3$. For $p = 2$, the scalars ζ in the identities $R_x = \zeta \chi_A$ have been determined by Marcelo–Shinoda [46, §4] and [18, §5]. Now assume that $p = 3$. For those cuspidal character sheaves A where the corresponding conjugacy class Σ is unipotent (there are three of them), the scalars ζ are also determined by [46, §4]. By [53, §7.2], the remaining four cuspidal character sheaves are analogous to those denoted above by A_2, A_5, A_6 and A_7 . One checks that the discussions in §7.7, 7.10, 7.11 can be applied almost verbatim to the case $p = 3$, and yield the same results. The Green functions for $p = 3$ are known by [46] (see also [19, §5]).

Acknowledgements

I am indebted to Jean Michel for much help with his programs [47], for independently verifying some of my computations, and for pointing out the reference Bonnafé [2]. Many thanks are due to Lacri Iancu and Gunter Malle for a careful reading of the manuscript, which led to the correction of several inaccuracies and to a number of improvements of the exposition. I am also grateful to an unknown referee for a number of useful comments. This article is a contribution to the SFB-TRR 195 “Symbolic Tools in Mathematics and their Application” by the DFG (Deutsche Forschungsgemeinschaft), Project-ID 286237555.

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