Dynkin Diagrams of Simple Lie Algebras

Cyclic Groups

Classical Steinberg Groups

Classical Chevalley Groups

The Tits group

Alternates

The Periodic Table Of Finite Simple Groups

The sporadic suzuki group is unrelated to the families of Suzuki groups.

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The greatest mathematical paper of all time

Euclid’s *Elements*, Newton’s *Principia*, . . . and in the past 200 years:


MathSciNet Review MR1007036 by Jean Dieudonné:

“Many mathematicians will disagree, […]. One may however observe that:

(1) nobody had tackled the problem before Killing;
(2) he solved it by methods he invented;
(3) nobody collaborated with him for that solution;
(4) the result became a most important milestone in modern mathematics.

I think it is not so easy to find papers exhibiting all those features”
Lie algebra = infinitesimal version of a transformation group

A vector space $L$ equipped with a product $x \cdot y$ such that

- $x \cdot x = 0$,
- $x \cdot (y \cdot z) + y \cdot (z \cdot x) + z \cdot (x \cdot y) = 0$ (Jacobi identity).

Usually write $x \cdot y$ as $[x, y]$ (Lie bracket).

**Example.** Vector product in $L = \mathbb{R}^3$:

![Vector product](image)

**Example.** Matrices: $L = M_n(\mathbb{C})$ with bracket $[A, B] = A \cdot B - B \cdot A$; denote $\mathfrak{gl}_n(\mathbb{C})$.

Subalgebra $\mathfrak{sl}_n(\mathbb{C}) = \{ A \in M_n(\mathbb{C}) \mid \text{Trace}(A) = 0 \}$. 
An infinite-dimensional example

Start with $R = \mathbb{C}[X, X^{-1}]$ (Laurent polynomials).

A linear map $D: R \to R$ is called a derivation if

$$D(f \cdot g) = f \cdot D(g) + D(f) \cdot g$$

for all $f, g \in R$.

Example: $D =$ usual (formal) derivative with respect to $X$.

$$D(X^n) = nX^{n-1}, \quad D(X^{-1}) = -X^{-2}, \quad \text{etc.}$$

Let $L =$ vector space of all derivations of $R$.

Exercise: If $D, D' \in L$, then $[D, D'] = D \circ D' - D' \circ D \in L$. So $L$ is a Lie algebra.

For $m \in \mathbb{Z}$, we set $L_m(f) = -X^{m+1} \cdot D(f)$ for $f \in R$. Then:

- $\{L_m \mid m \in \mathbb{Z}\}$ is a vector space basis of $L$.
- $[L_m, L_n] = (m - n)L_{m+n}$ for all $m, n \in \mathbb{Z}$.

$L$ is called “Witt algebra” $\Rightarrow$ important in mathematical physics.
A subspace $U \subseteq L$ is called an ideal if $[u, x] \in U$ and $[x, u] \in U$ for all $x \in L, u \in U$. In this case, $\bar{L} = L/U$ also is a Lie algebra. So $L$ “built up” from $U$ and $\bar{L}$.

**Definition.** $L$ is called simple if $L \neq \{0\}$, the bracket is not identically zero and there is no proper ideal in $L$.

**Cartan–Killing (~1890):** The finite-dimensional simple Lie algebras over $\mathbb{C}$ are classified by “Dynkin diagrams”.

```
A_n  B_n  C_n  D_n  E_6  E_7  E_8  F_4  G_2
```

[Diagrams of Dynkin diagrams for $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$]
Infinite families: Lie algebras of matrices

$A_n \leftrightarrow \mathfrak{sl}_{n+1}(\mathbb{C})$, $B_n \leftrightarrow \mathfrak{so}_{2n+1}(\mathbb{C})$, $C_n \leftrightarrow \mathfrak{sp}_{2n}(\mathbb{C})$, $D_n \leftrightarrow \mathfrak{so}_{2n}(\mathbb{C})$.

Exceptional algebras:

$\dim \mathfrak{g}_2 = 14$, $\dim \mathfrak{f}_4 = 52$, $\dim \mathfrak{e}_6 = 78$, $\dim \mathfrak{e}_7 = 133$, $\dim \mathfrak{e}_8 = 248$.

S. Garibaldi, $E_8$, the most exceptional group. Bull. AMS 53 (2016), 643–671.

“The Lie algebra $\mathfrak{e}_8$ or Lie group $E_8$ was first sighted by a human being sometime in summer or early fall of 1887, by Wilhelm Killing as part of his program to classify the semisimple finite-dimensional Lie algebras over the complex numbers. [...]”

Since then, it has been a source of fascination for mathematicians and others in its role as the largest of the exceptional Lie algebras. [...]”

If we know some statement for all groups except $E_8$, then we do not really know it.”
How does a Dynkin diagram determine a simple Lie algebra?

If diagram has nodes 1, \ldots, n, then form “Cartan matrix” \( A = (a_{ij})_{1 \leq i, j \leq n} \) where:

- \( a_{ii} = 2 \) for all \( i \);
- \( a_{ij} = 0 \) if \( i \neq j \) are not joined by an edge;
- if \( i \neq j \) are joined by a single edge, set \( a_{ij} = a_{ji} = -1 \);
- if \( i \neq j \) are joined by \( m \geq 2 \) edges (arrow towards \( j \)), set \( a_{ij} = -1, a_{ji} = -m \).

\[
B_3 \quad \begin{array}{ccc}
1 & 2 & 3 \\
\end{array} \quad \sim \quad A = \begin{pmatrix}
2 & -2 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}
\]

Chevalley generators (“épinglage”) of corresponding Lie algebra \( L \):

- \( L = \langle e_i, f_i, h_i \mid 1 \leq i \leq n \rangle_{\text{Lie}} \),
- \( [e_i, f_i] = h_i, \quad [e_i, f_j] = 0 \) (if \( i \neq j \)), \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ji}f_j.
- \( H := \langle h_1, \ldots, h_n \rangle_\mathbb{C} \) “Cartan” subalgebra of \( L \).

This determines \( L \) up to isomorphism.
Vector space basis of $L$

Consider $\mathbb{R}^n$ with standard basis $\{\alpha_1, \ldots, \alpha_n\}$. Define

$$s_i: \mathbb{R}^n \to \mathbb{R}^n, \quad \alpha_j \mapsto \alpha_j - a_{ij}\alpha_i.$$ 

Then $s_i^2 = \text{id}_n$ and $W = \langle s_1, \ldots, s_n \rangle \subseteq \text{GL}_n(\mathbb{R})$ is a finite reflection group.

$$\Phi := \{w(\alpha_i) \mid w \in W, 1 \leq i \leq n\}$$ is the abstract “root system” associated with $A$.

Corresponding Lie algebra $L$ has basis $\{h_1, \ldots, h_n\} \cup \{e_\alpha \mid \alpha \in \Phi\}$.

“Root strings” (Killing): Let $\alpha, \beta \in \Phi$, $\alpha \neq \pm \beta$.

Let $p = p_{\alpha, \beta} \geq 0$ and $q = q_{\alpha, \beta} \geq 0$ be maximal such that

$$\beta - q\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \ldots, \beta + p\alpha \in \Phi.$$

C. Chevalley (1955): $e_\alpha$ can be chosen such that $e_{\alpha_i} = \pm e_i$, $e_{-\alpha_i} = \pm f_i$ and

$$[e_\alpha, e_\beta] = \pm(q_{\alpha, \beta} + 1)e_{\alpha+\beta}$$ whenever $\alpha, \beta, \alpha + \beta \in \Phi$

(and with this normalization, the $e_\alpha$ are unique up to sign).
An aside: root systems and lattices

Let \( \Gamma := \) all \( \mathbb{Z} \)-linear combinations of roots \( \alpha \in \Phi \). Then \( \Gamma \) is a “lattice” in \( \mathbb{R}^n \).

Conversely, let \( \Gamma \subseteq \mathbb{R}^n \) be a lattice.

- \( \Gamma \) is integral if \( (x, y) \in \mathbb{Z} \) for \( x, y \in \Gamma \); then \( \Gamma \) is even if \( (x, x) \in 2\mathbb{Z} \) for \( x \in \Gamma \).
- Dual lattice \( \Gamma^* := \{ x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in \Gamma \} \); if \( \Gamma \) integral, then \( \Gamma \subseteq \Gamma^* \).
- \( \Gamma \) is unimodular if \( \Gamma \) is integral and \( \Gamma = \Gamma^* \).

Up to isomorphism, there is only one even unimodular lattice in \( \mathbb{R}^8 \).

(History goes back to around 1870, before Lie and Killing.) This is given by

\[
\Gamma_8 := \left\{ (x_i) \in \mathbb{R}^8 \mid 2x_i \in \mathbb{Z}, \ x_i - x_j \in \mathbb{Z}, \ \sum_i x_i \in 2\mathbb{Z} \right\}.
\]

The set \( R := \{ x \in \Gamma_8 \mid (x, x) = 2 \} \) contains precisely 240 vectors; these form the “root system” associated with the Dynkin diagram \( E_8 \).

(A further famous example in dimension 24: the Leech lattice \( \leadsto \) Conway's finite simple groups. See, e.g., J-P. Serre, A Course in Arithmetic, or W. Ebeling, Lattices and Codes, Springer-Verlag.)

Simple Lie algebra $L$ + field $k$ $\leadsto$ group $G_L(k)$.

(Mimic exponential “$\exp(e^\alpha)$” over arbitrary field $k$; then $G_L(k) = \langle \exp(te^\alpha) \mid \alpha \in \Phi, t \in k \rangle$.)

- $k$ algebraically closed: $G_L(k)$ simple algebraic group over $k$,
  $\SL_n(k), \quad \Sp_{2n}(k), \quad \SO_n(k), \quad E_8(k), \quad \ldots$

- $k$ finite $\leadsto$ new families (at the time) of finite simple groups,
  e.g., $|E_8(\mathbb{F}_q)| = q^{248} + \text{lower powers of } q$ (where $q$ is a prime power).

Textbook references:

- BOURBAKI, Groupes et algèbres de Lie, 1968/1975;
- STEINBERG, Lectures on Chevalley groups, 1967/68 (now available from AMS);
- HUMPHREYS, Introduction to Lie algebras and representation theory, 1972;
- CARTER, Simple groups of Lie type, 1972;
So what are the news about Lie algebras?

George Lusztig, arXiv:1309.1382 (3 pages):

“The Lie group of type $E_8$ can be obtained from the graph $E_8$ by a method of Chevalley (1955), simplified using theory of canonical basis (1990).”

⇝ 3 more papers: one by Lusztig, two by myself.
Why look for a simplification?

- Construction of $L$ is an issue, especially for exceptional types.
- Chevalley’s construction of $G_L(k)$ relies on choice of $e_\alpha$’s.

J. Tits 1966: Start with vector space $M$ of correct dimension, fix basis $\{h_1, \ldots, h_n\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ and try to define Lie bracket for basis elements. Relies on systematic study of sign choices in Chevalley’s normalisation of $e_\alpha$.

J-P. Serre 1966: Start with free Lie algebra generators $\{e_i, f_i \mid 1 \leq i \leq n\}$ and add defining relations (nowadays called “Serre relations”) to obtain a presentation. Very elegant, does not resolve issue of making choices for the $e_\alpha$’s.

C.-M. Ringel 1990: Fix orientation of Dynkin diagram and use the representation theory of quivers and Hall polynomials. Then $e_\alpha$’s correspond to well-defined indecomposable objects in certain category of modules. Pre-cursor of “canonical bases”, still relies on $2^{n-1}$ choices of orientations.
The simplified construction

Recall “root strings”: Let $\alpha, \beta \in \Phi$, $\alpha \neq \pm \beta$. Then

$$p_{\alpha,\beta} := \max \{i \geq 0 \mid \beta + i\alpha \in \Phi\} \quad \text{and} \quad q_{\alpha,\beta} := \max \{j \geq 0 \mid \beta - j\alpha \in \Phi\}.$$ 

Let $M$ be a vector space over $\mathbb{C}$ of the “correct” dimension, with a given basis $\{u_1, \ldots, u_n\} \cup \{v_\alpha \mid \alpha \in \Phi\}$. Define linear maps $e_i, f_i : M \to M$ by

$$e_i(u_j) := |a_{ji}| v_{\alpha_i}, \quad e_i(v_\alpha) := \begin{cases} (q_{\alpha_i,\alpha} + 1)v_{\alpha+\alpha_i} & \text{if } \alpha + \alpha_i \in \Phi, \\ u_i & \text{if } \alpha = -\alpha_i, \\ 0 & \text{otherwise}; \end{cases}$$

$$f_i(u_j) := |a_{ji}| v_{-\alpha_i}, \quad f_i(v_\alpha) := \begin{cases} (p_{\alpha_i,\alpha} + 1)v_{\alpha-\alpha_i} & \text{if } \alpha - \alpha_i \in \Phi, \\ u_i & \text{if } \alpha = \alpha_i, \\ 0 & \text{otherwise}. \end{cases}$$

(Note: Matrices of $e_i, f_i$ have entries in $\mathbb{Z}_{\geq 0}$ — in fact, in $\{0, 1, 2, 3\}$.)


Consider the Lie algebra $\mathfrak{gl}(M)$ and let $L := \langle e_i, f_i \mid 1 \leq i \leq n \rangle_{\text{Lie}} \subseteq \mathfrak{gl}(M)$. Then $L$ is a simple Lie algebra corresponding to the given Dynkin diagram.
Example: Type $G_2$ with Cartan matrix $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

- $\mathbb{R}^2$ with standard basis $\{\alpha_1, \alpha_2\}$. Matrices $s_1, s_2 \in \text{GL}_2(\mathbb{R})$ given by
  
  $$s_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}.$$ 

- Applying $s_1, s_2$ repeatedly to $\alpha_1 = (1,0), \alpha_2 = (0,1)$, we obtain
  
  $$\Phi = \{\pm(1,0), \pm(0,1), \pm(1,1), \pm(1,2), \pm(1,3), \pm(2,3)\}.$$ 

- Simple Lie algebra $L = \langle e_1, e_2, f_1, f_2 \rangle_{\text{Lie}} \subseteq \mathfrak{gl}_{14}(\mathbb{C})$. For example:


- $e_1, e_2$ upper triangular
- $f_1, f_2$ lower triangular

(A dot stands for 0)
Resolving sign issue in Chevalley’s basis

We can transport the basis \{u_1, \ldots, u_n\} \cup \{v_\alpha \mid \alpha \in \Phi\} from \(M\) to \(L\) and obtain:

**Corollary.**

- There is a basis \{\tilde{h}_1, \ldots, \tilde{h}_n\} \cup \{e_\alpha \mid \alpha \in \Phi\} of \(L\) such that:

  \[
  [e_i, \tilde{h}_j] = |a_{ji}| e_{\alpha_i}, \quad [e_i, e_\alpha] = \begin{cases} 
  (q_{\alpha_i,\alpha} + 1) e_{\alpha+\alpha_i} & \text{if } \alpha + \alpha_i \in \Phi, \\
  \tilde{h}_j & \text{if } \alpha = -\alpha_i, \\
  0 & \text{otherwise}; 
  \end{cases}
  \]

  \[
  [f_i, \tilde{h}_j] = |a_{ji}| e_{-\alpha_i}, \quad [f_i, e_\alpha] = \begin{cases} 
  (p_{\alpha_i,\alpha} + 1) e_{\alpha-\alpha_i} & \text{if } \alpha - \alpha_i \in \Phi, \\
  \tilde{h}_j & \text{if } \alpha = \alpha_i, \\
  0 & \text{otherwise}. 
  \end{cases}
  \]

- This basis of \(L\) is unique up to a global constant.

- In particular, the structure constants in the equations \([e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}\) (for \(\alpha, \beta, \alpha + \beta \in \Phi\)) are uniquely determined up to a global constant.
Where do the formulae for the action of $e_i, f_i$ come from?

- ≈ 1985: Introduction of quantised enveloping algebras of Lie algebras (Drinfeld, Jimbo), as Hopf algebra deformations depending on a parameter $\nu$. Origins in mathematical physics, quantum integrable systems, ...

- Early 1990’s: Discovery of “canonical bases” (Lusztig) and “crystal bases” (Kashiwara). Specialisation $\nu \mapsto 1$ gives rise to canonical bases in irreducible representations of ordinary simple Lie algebras.

**Lusztig (1990 + J. Comb. Algebra 2017).** Consider $L \hookrightarrow \mathfrak{gl}(M)$, as above. This is an irreducible representation and $\{u_1, \ldots, u_n\} \cup \{v_\alpha \mid \alpha \in \Phi\}$ is the “canonical basis” of $M$ in the above sense. (≈ Explains why formulae are “natural”.)

**Explicit/simple construction of $L$ and, hence, also of $GL(k)$, useful for:**

- algorithmic problems: nilpotent orbits, matrix group recognition project, …;

- teaching courses on Lie algebras (“existence theorem”).
My motivation for studying Lie algebras and Chevalley groups

**Group theory = study of symmetries**

- Continuous $\leadsto$ Lie groups
- Discrete $\leadsto$ finite groups

“Atoms” of symmetry: finite **simple** groups.

A highlight of 20th century mathematics: **Classification**.
(first announced 1981 • completed 2004: Aschbacher, Smith • 12000 pages of proof)

The Periodic Table Of Finite Simple Groups

Dykin Diagrams of Simple Lie Algebras

<table>
<thead>
<tr>
<th>A_1, A_2, F_4</th>
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<tr>
<td>A_3, A_4, G_2</td>
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<td>168</td>
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<tr>
<td>A_5, A_6, D_6</td>
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<td>584</td>
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<td>A_9, A_10, D_5</td>
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| C_2 | 2 |
| C_3 | 3 |
| C_4 | 5 |
| C_5 | 7 |
| C_6 | 11 |
| C_7 | 13 |
| C_8 | 17 |
| C_9 | 19 |
| C_{10} | p |

<table>
<thead>
<tr>
<th>Alternating Groups</th>
<th>Classical Chevalley Groups</th>
<th>Exceptional Groups</th>
<th>Classical Steinberg Groups</th>
<th>Twisted Groups</th>
<th>Suzuki Groups</th>
<th>Stein Groups and Tits Groups</th>
<th>Simple Groups</th>
<th>Sporadic Groups</th>
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<td>E_{7}</td>
<td>E_{8}</td>
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The groups shown in the upper left are other names by which they
are known. The groups in parentheses are isomorphic to
the groups in the table except for those with an "i".

This group table is a snapshot of the current state of knowledge, with
many groups still to be discovered. New groups may be found in the future,
and the list of known groups may expand

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