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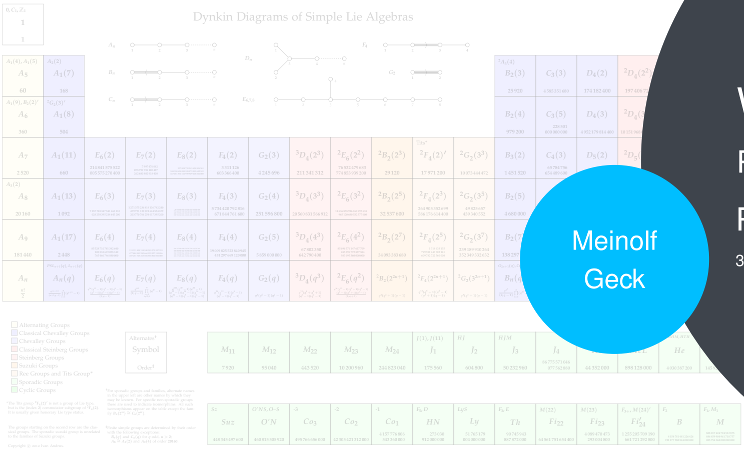

What is bad about bad primes?

RepNet Virtual Seminar

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k	an algebraically closed field, mostly $k = \overline{\mathbb{F}}_p$ where p is a prime;
G	a simple algebraic group over k, e.g. $\mathrm{SL}_n(k)$, $\mathrm{Sp}_{2n}(k)$, $\ldots$, $E_8(k)$;
$\mathfrak{g}$	the Lie algebra of G.

Main subjects of the talk.

- Unipotent conjugacy classes in G and nilpotent orbits in $\mathfrak{g}$.
- (Representations of the finite groups $G(\mathbb{F}_q)$, where q is a power of p.)
- Model case for use of computer algebra methods.

All based on my two recent articles in Transformation Groups (online May 2020) and the Journal of Software for Algebra and Geometry (2020).

Some background/motivation.

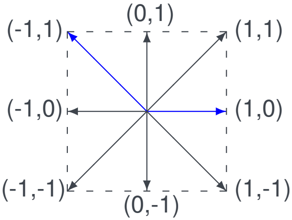
- For $p \gg 0$, the classification of unipotent classes of G over $k = \overline{\mathbb{F}}_p$ is the same as in char 0. Eventually: p not "bad" sufficient (... Premet, J. Algebra 2003).
- Kawanaka: "GGGRs", Arcata Proceedings (1987).
  For each unipotent $u \in G(\mathbb{F}_q)$, where $q = p^f$, define character of $G(\mathbb{F}_q)$ induced from a linear character of a certain unipotent subgroup.
  - ◇ Works if p is not a "bad" prime for G using features from char 0 theory.
  - ◇ GGGRs are characters of ℓ-modular projective representations where $\ell \neq p$.
  - ◇ Proof of Kawanaka's conjecture by Brunat–Dudas–Taylor, Annals of Math., to appear $\rightsquigarrow$ Unitriangularity of ℓ-modular decomposition matrix of $G(\mathbb{F}_q)$.
- Lusztig "Unipotent elements in small characteristics I–IV" (2005–2011).
  - ◇ Present a picture about unipotent classes which applies for arbitrary p and is as close as possible to char 0.
  - ◇ Is it possible to define GGGRs for $G(\mathbb{F}_q)$ when q is a power of a "bad" prime ?

Definition. Borel et al., Springer Lecture Notes 131 (1970)

Let Φ be the root system of G and $\Delta = \{\alpha_i \mid i \in I\}$ a set of simple roots. So every $\alpha \in \Phi$ can be written uniquely as $\alpha = \sum_{i \in I} n_i \alpha_i$ with all $n_i \in \mathbb{Z}$ of the same sign.

The prime p is called a bad prime for G if $p = n_i$ for some $i \in I$ and some α.

Example: $G = \mathrm{Sp}_4(k)$ has root system of type C_2.



$\Delta = \{\alpha_1 = (1, 0), \alpha_2 = (-1, 1)\}$

$(0, 1) = \alpha_1 + \alpha_2$,

$(1, 1) = 2\alpha_1 + \alpha_2$, etc.

$\rightsquigarrow$ $p = 2$ only bad prime

For the various types of G, the bad primes are as follows.

$$\begin{aligned} A_n &: \quad \text{none}; \\ B_n, C_n, D_n &: \quad p = 2; \\ G_2, F_4, E_6, E_7 &: \quad p = 2, 3; \\ E_8 &: \quad p = 2, 3, 5. \end{aligned}$$

Bad primes often appear as trouble-makers in the theory of G and $\mathfrak{g}$.

Example from Borel et al. (and relevant for us):

If $k = \overline{\mathbb{F}}_p$ and p is not bad for G, then the number of unipotent classes of G is finite, and there is a bijection $\{$unipotent classes of $G\} \stackrel{1-1}{\longleftrightarrow} \{$nilpotent orbits in $\mathfrak{g}\}$.

Argument for finiteness in Borel et al. does not work in bad characteristic; in general, no bijection as above if p is bad.

Extension of “finiteness” to bad primes by Lusztig (1976), using the “Deligne–Lusztig theory” of characters of the finite groups $G(\mathbb{F}_{p^f}) \subseteq G$ for $f = 1, 2, 3, \ldots$

Let $k = \overline{\mathbb{F}}_p$ and $\mathcal{U}_p$ be the (finite!) set of unipotent classes of G.

Partial order on $\mathcal{U}_p$: $\mathcal{O}_1 \leqslant \mathcal{O}_2$ if $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$.

G_0 := group of the same type as G but over $\mathbb{C}$; $\mathcal{U}_0 := \{$unipotent classes of $G_0\}$.

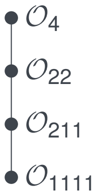
Theorem. Spaltenstein, Springer Lecture Notes 946 (1982)

There is a unique injective map $\pi\colon \mathcal{U}_0 \hookrightarrow \mathcal{U}_p$ that preserves the dimension of classes, is compatible with parabolic subgroups and an isomorphism of partially ordered sets onto its image. If p is not bad for G, then π is a bijection.

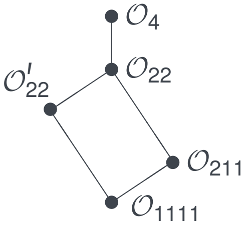
(Proof uses explicitly known classification of unipotent classes in all cases.)

Example: $G = \mathrm{Sp}_4(k)$.

$\mathcal{U}_0$:



$\mathcal{U}_2$:



Thus, it makes sense to speak of unipotent classes of G that "come from char 0".

These are classified by Dynkin–Kostant theory (see Carter's book on finite groups of Lie type): Let $\mathcal{O} \in \mathcal{U}_p$ come from char 0. Then there is a corresponding "weighted Dynkin diagram", that is, a function $d\colon \Phi \to \mathbb{Z}$ such that

- $d(-\alpha) = -d(\alpha)$ and $d(\alpha + \beta) = d(\alpha) + d(\beta)$ if $\alpha, \beta, \alpha + \beta \in \Phi$;
- $d(\alpha_i) \in \{0, 1, 2\}$ where $\Delta = \{\alpha_i \mid i \in I\}$ set of simple roots.

Consider Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$. Set

$$\mathfrak{g}_d(0) = \mathfrak{h} \oplus \bigoplus_{\alpha\in d^{-1}(0)} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{g}_d(n) = \bigoplus_{\alpha\in d^{-1}(n)} \mathfrak{g}_\alpha \quad \text{for } n \neq 0.$$

$\rightsquigarrow$ Grading $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_d(n)$ with $[\mathfrak{g}_d(n), \mathfrak{g}_d(m)] \subseteq \mathfrak{g}_d(n+m)$.

- Assume p is not bad. Recall $\mathcal{U}_p \stackrel{1-1}{\longleftrightarrow} \{\text{nilpotent orbits in } \mathfrak{g}\}$. The nilpotent orbit in bijection with $\mathcal{O}$ intersects $\mathfrak{g}_d(2)$ in a (well-defined) dense open set.
- Tables of weighted Dynkin diagrams in the appendix of Carter's book.

A key step in Kawanaka's construction of GGGRs.

Assume p is not bad. Consider above grading $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_d(n)$, for some $\mathcal{O}$.

Recall that the nilpotent orbit $\leftrightarrow \mathcal{O}$ intersects $\mathfrak{g}_d(2)$ in a dense open set.

Dually, there is a dense open set of linear functions $\lambda\colon \mathfrak{g}_d(2) \to k$ such that

$$\sigma_\lambda\colon \mathfrak{g}_d(1) \times \mathfrak{g}_d(1) \to k, \qquad (y, z) \mapsto \lambda([y, z]),$$

is a non-degenerate, symplectic form.

[GGGRs: Find maximal isotropic subspace $\mathfrak{g}_d(1.5) \subseteq \mathfrak{g}_d(1)$. Then $\mathfrak{g}_d(1.5) \oplus \bigoplus_{n\geqslant 2} \mathfrak{g}_d(n)$ is the Lie algebra of a connected unipotent subgroup $U_{\geqslant 1.5} \subseteq G$. From λ one obtains a homomorphism $\chi_\lambda\colon U_{\geqslant 1.5}(\mathbb{F}_q) \to \mathbb{C}^\times$. Inducing χ_λ from $U_{1.5}(\mathbb{F}_q)$ to $G(\mathbb{F}_q)$ yields GGGRs associated with $\mathcal{O}$.]

Plan/Problem/Question: Make no assumption on p. Consider $\mathcal{O} \in \mathcal{U}_p$ coming from char 0, with weighted Dynkin diagram $d\colon \Phi \to \mathbb{Z}$. Still obtain grading $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_d(n)$. Investigate if, or when there are $\lambda\colon \mathfrak{g}_d(2) \to k$ such that

$$\sigma_\lambda\colon \mathfrak{g}_d(1) \times \mathfrak{g}_d(1) \to k, \quad (y, z) \mapsto \lambda([y, z]), \quad \text{is non-degenerate?}$$

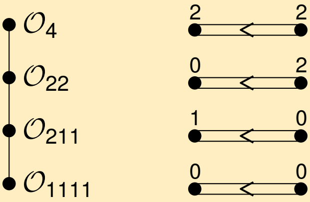
Let $\mathfrak{g}_0$:= Lie algebra of G_0 (over $\mathbb{C}$). Fix Chevalley basis

$$\{h_i \mid i \in I\} \cup \{e_\alpha \mid \alpha \in \Phi\} \qquad \text{(canonical choice by G., Proc. AMS 2017)}$$

This spans Lie algebra $\mathfrak{g}_\mathbb{Z}$ over $\mathbb{Z}$ such that $\mathfrak{g} = k \otimes_\mathbb{Z} \mathfrak{g}_\mathbb{Z}$ and $\mathfrak{g}_\alpha = \langle \bar{e}_\alpha \rangle_k$ for $\alpha \in \Phi$.

Example: $G = \mathrm{Sp}_4(k)$.

$\mathcal{U}_0$:



Consider $\mathcal{O}_{211}$. We have $\mathfrak{g}_d(1) = \langle \bar{e}_{\alpha_1}, \bar{e}_{\alpha_1+\alpha_2} \rangle_k$ and $\mathfrak{g}_d(2) = \langle \bar{e}_{2\alpha_1+\alpha_2} \rangle_k$, furthermore, $[\bar{e}_{\alpha_1}, \bar{e}_{\alpha_1+\alpha_2}] = 2\bar{e}_{2\alpha_1+\alpha_2}$. So Gram matrix of σ_λ given by

$$\begin{pmatrix} 0 & 2x_1 \\ -2x_1 & 0 \end{pmatrix} \qquad \text{where} \qquad x_1 := \lambda(\bar{e}_{2\alpha_1+\alpha_2}) \in k.$$

Determinant is $4x_1^2$. So, if $p = 2$, then σ_λ is always degenerate; if $p \neq 2$, obtain non-degenerate σ_λ for $x_1 \neq 0$ ("open" condition). — In any case, nothing new !

Example: $G = F_4(k)$. Let $\mathcal{O} = \tilde{A}_1$ with weighted diagram d : $0 — 0 \Rightarrow 0 — 1$

Then $\dim \mathfrak{g}_d(1) = 8$ and $\dim \mathfrak{g}_d(2) = 7$. So the Gram matrix of σ_λ has size 8×8:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -2x_1 & -2x_2 & -2x_3 & -x_4 \\ 0 & 0 & 2x_1 & 2x_2 & 0 & 0 & -x_4 & -2x_5 \\ 0 & -2x_1 & 0 & 2x_3 & 0 & x_4 & 0 & -2x_6 \\ 0 & -2x_2 & -2x_3 & 0 & -x_4 & 0 & 0 & -2x_7 \\ 2x_1 & 0 & 0 & x_4 & 0 & 2x_5 & 2x_6 & 0 \\ 2x_2 & 0 & -x_4 & 0 & -2x_5 & 0 & 2x_7 & 0 \\ 2x_3 & x_4 & 0 & 0 & -2x_6 & -2x_7 & 0 & 0 \\ x_4 & 2x_5 & 2x_6 & 2x_7 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $x_1, \ldots, x_7 \in k$ are the values of λ on the 7 basis elements of $\mathfrak{g}_d(2)$.
We can just notice that, if we set $x_4 := 1$ and $x_i := 0$ for all $i \neq 4$, then $\det = \pm 1$.
Then σ_λ will be non-degenerate for this choice of λ, and this works for any field k.

So this is a new case, where it might be possible to define GGGRs in char 2 and 3.

Use computer algebra package to perform these computations:

```
####################################################################
##    Welcome to  version 1.1  of the Julia module  ‘ChevLie‘:    ##
##       CONSTRUCTING  LIE  ALGEBRAS  AND  CHEVALLEY  GROUPS      ##
##    https://pnp.mathematik.uni-stuttgart.de/iaz/iaz2/geckmf/    ##
##    Type   ?LieAlg     for first help; all comments welcome!    ##
####################################################################
```

- Systematically worked out all Gram matrices for G of exceptional type.
- Some determinants very hard to compute. Use pfaffian or Groebner bases techniques to solve such cases (suggestion by U. Thiel and A. Steel).
- By inspection of all data, obtain uniform characterisation of possible new cases for GGGRs in terms of “unipotent support” of characters of $G(\mathbb{F}_q)$.
- Also new characterisation of Lusztig’s “special” unipotent classes.

First ingredient for "special" unipotent classes:

W Weyl group of G, $\mathrm{Irr}(W)$ complex irreducible characters.

Lusztig 1978/1979: Subset $\mathcal{S} \subseteq \mathrm{Irr}(W)$ of "special" characters.

- There is a partition $\mathrm{Irr}(W) = \bigsqcup_{\mathcal{F}} \mathrm{Irr}(W \mid \mathcal{F})$, where $\mathcal{F}$ runs over the two-sided Kazhdan–Lusztig cells of W. Each $\mathrm{Irr}(W \mid \mathcal{F})$ contains a unique element of $\mathcal{S}$.
- For $\phi \in \mathrm{Irr}(W)$ define b_ϕ := smallest $i \geqslant 0$ such that ϕ appears in i-th symmetric power of reflection character. For a two-sided cell $\mathcal{F}$, there is a unique $\phi_0 \in \mathrm{Irr}(W \mid \mathcal{F})$ where b_ϕ reaches its minimum; then $\phi_0 \in \mathcal{S}$.
- $\mathcal{S} \subseteq \mathrm{Irr}(W)$ explicity known in all cases (see appendix of Carter's book).
- Lusztig 2018: New characterisation of $\mathcal{S}$ in terms of a "positivity" property.

Examples: W of type A_{n-1} (symmetric group) $\Rightarrow$ $\mathcal{S} = \mathrm{Irr}(W)$.

W of type C_2 (dihedral of order 8) $\Rightarrow$ $\mathcal{S} = \{1_W, \varepsilon_W, 2\text{–dim.}\} \subsetneq \mathrm{Irr}(W)$.

W of type E_8 $\Rightarrow$ $|\mathcal{S}| = 46$ and $|\mathrm{Irr}(W)| = 112$.

Second ingredient for “special” unipotent classes: Springer correspondence.

Consider $G(\mathbb{F}_q)$, where $q = p^f$. For $w \in W$ let R_w be the virtual character defined by Deligne and Lusztig (1976). For $\phi \in \mathrm{Irr}(W)$, set

$$R_\phi := \frac{1}{|W|} \sum_{w \in W} \phi(w) R_w \qquad \text{“almost character”.}$$

Then there is a unique unipotent class $\mathcal{O}_\phi$ such that

$$\{g \in G(\mathbb{F}_q) \text{ unipotent} \mid R_\phi(g) \neq 0\} \subseteq \overline{\mathcal{O}}_\phi \qquad \text{and} \qquad R_\phi|_{\mathcal{O}_\phi(\mathbb{F}_q)} \neq 0.$$

This map $\phi \mapsto \mathcal{O}_\phi$ is the Springer correspondence (doesn’t depend on $q = p^f$).

The unipotent class $\mathcal{O} \in \mathcal{U}_p$ is called “special” if $\mathcal{O} = \mathcal{O}_\phi$ for some $\phi \in \mathcal{S}$.

Lusztig, Notes on unipotent classes (1997):

“The special unipotent classes play a key role in several problems in representation theory. [...] Unfortunately, their definition is totally un-geometrical. For this reason, special unipotent classes are often regarded as rather mysterious objects.”

- Special unipotent classes come from char 0 and are independent of p.

There is a well-defined subset $\mathcal{U}_0^* \subseteq \mathcal{U}_0$ such that, for any p, the special unipotent classes in $\mathcal{U}_p$ correspond to $\mathcal{U}_0^*$ under Spaltenstein's embedding $\mathcal{U}_0 \hookrightarrow \mathcal{U}_p$.

[Follows from explicit knowledge of $\mathcal{S} \subseteq \mathrm{Irr}(W)$ and the Springer correspondence in all cases; see Carter's book and Lusztig–Spaltenstein, Spaltenstein, 1985.]

- There is an order-reversing involution on the set of special unipotent classes.

This is induced by the involution on $\mathrm{Irr}(W \mid \mathcal{F})$ given by tensoring with the sign character. See explicit diagrams in Carter's book.

- Is it possible to characterise the subset $\mathcal{U}_0^* \subseteq \mathcal{U}_0$ "less mysteriously" ?
  Note: This is now a question about unipotent classes of G_0 (group over $\mathbb{C}$).

Back to “classical” Dynkin–Kostant setting, where $\mathcal{U}_0$ = {unipotent classes of G_0} is classified by weighted Dynkin diagrams. Let again $\mathfrak{g}_0 :=$ Lie algebra of G_0.

Fix Chevalley basis $\{h_i \mid i \in I\} \cup \{e_\alpha \mid \alpha \in \Phi\}$. This spans Lie algebra $\mathfrak{g}_\mathbb{Z}$ over $\mathbb{Z}$.

Now fix $\mathcal{O} \in \mathcal{U}_0$ and corresponding d. $\rightsquigarrow$ Grading $\mathfrak{g}_\mathbb{Z} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\mathbb{Z},d}(n)$.

In particular, $\mathfrak{g}_{\mathbb{Z},d}(n) = \langle e_\alpha \mid d(\alpha) = n\rangle_\mathbb{Z}$ for $n = 1, 2$.

Definition. Integrality condition, see G. (2020)

We say that d is $\mathbb{Z}$-special if there is some $\lambda \in \mathrm{Hom}(\mathfrak{g}_{\mathbb{Z},d}(2), \mathbb{Z})$ such that

$$\sigma_\lambda \colon \mathfrak{g}_{\mathbb{Z},d}(1) \times \mathfrak{g}_{\mathbb{Z},d}(1) \to \mathbb{Z}, \qquad (y, z) \mapsto \lambda([y, z]),$$

is non-degenerate over $\mathbb{Z}$. (If $\mathfrak{g}_{\mathbb{Z},d}(1) = \{0\}$, then d is declared $\mathbb{Z}$-special.)

Conjecture. $\mathcal{O}$ is special if and only if d is $\mathbb{Z}$-special.

- Using our computations of Gram matrices, true for all G_0 of exceptional types.
- For G_0 of classical type, proof by Dong and Yang, `arxiv:1910.03764`.
- Relevance for real Lie groups: see Vogan’s MIT Virtual Lie Group Seminar talk “What’s special about special?” at `http://www-math.mit.edu/~dav/LG/`.