Abstract methods for constructing t-structures (talk notes)

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July 7, 2011

Abstract

Notes for the talk at Workshop on t-structures and related topics, July 7–9, 2011, Stuttgart. Report on joint work with M. Saorín, based on arXiv:1005.3248v2.

Contents

1 t-structures, weight structures and semiorthogonal decompositions

Throughout, let $\mathcal{T}$ be a triangulated category. We recall 3 definitions, formulated in a slightly non-classical way in order to stress the formal similarity among them:

t-structure: A pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories in $\mathcal{T}$ such that

- $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{X}^{\perp} = \mathcal{Y}$,
- $\mathcal{X}[1] \subseteq \mathcal{X}$ (and so $\mathcal{Y}[-1] \subseteq \mathcal{Y}$),
- approximation triangles exist (see below).

Motivation: $\mathcal{H} = \mathcal{X} \cap \mathcal{Y}[1]$ is an abelian subcategory of $\mathcal{T}$.

Semiorthogonal decomposition: A pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories in $\mathcal{T}$ such that

- $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{X}^{\perp} = \mathcal{Y}$,
- $\mathcal{X}[1] = \mathcal{X}$ (and so $\mathcal{Y}[-1] = \mathcal{Y}$),
- approximation triangles exist (see below).

Motivation: There are triangle equivalences $\mathcal{T}/\mathcal{X} \xrightarrow{\sim} \mathcal{Y}$ (Bousfield localization) and $\mathcal{T}/\mathcal{Y} \xrightarrow{\sim} \mathcal{X}$ (Bousfield colocalization).

Weight structure: A pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories in $\mathcal{T}$ such that

- $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{X}^{\perp} = \mathcal{Y}$,
- $\mathcal{X}[-1] \subseteq \mathcal{X}$ (and so $\mathcal{Y}[1] \subseteq \mathcal{Y}$),
- approximation triangles exist (see below).

Motivation from motivic theory, more details in other talks.

Difference only in the closure properties of $\mathcal{X}$ and $\mathcal{Y}$. The following crucial concept is involved in all the definitions:

Approximation triangles: We insist that each $U \in \mathcal{T}$ admits a triangle $X \to U \to Y \to X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Question. *Given a* **Hom-orthogonal pair** $(\mathcal{X}, \mathcal{Y})$*, that is a pair of full subcategories of* $\mathcal{T}$ *such that* $\mathcal{X} = {}^{\perp}\mathcal{Y}$ *and* $\mathcal{X}^{\perp} = \mathcal{Y}$*, when do approximation triangles exist?*

2 Frobenius exact categories

We address the question for algebraic triangulated categories, i.e. those obtained from Frobenius exact categories. Recall:

Exact category: An additive category $\mathcal{C}$ together with a designated class of kernel-cokernel pairs

$$\{0 \longrightarrow K_\lambda \xrightarrow{i_\lambda} L_\lambda \xrightarrow{d_\lambda} M_\lambda \longrightarrow 0 \mid \lambda \in \Lambda\},$$

called **conflations**, satisfying suitable axioms. Terminology:

inflation: The map i_λ in a conflation.

deflation: The map d_λ in a conflation.

Remark. If $\mathcal{A}$ is abelian, $\mathcal{C} \subseteq \mathcal{A}$ is an extension closed full subcategory, and the conflations in $\mathcal{C}$ are defined by

$$\{0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0 \text{ exact in } \mathcal{A} \mid K, L, M \in \mathcal{C}\},$$

we get an exact category. Every small exact category is of this form.

Given an exact category $\mathcal{C}$ and $K, M \in \mathcal{C}$, we can define Yoneda Ext:

$$\mathrm{Ext}^1_{\mathcal{C}}(M, K) = \{0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0 \text{ conflation}\}/\sim$$

and using that, we can define projective and injective objects.

Frobenius exact category: An exact category $\mathcal{C}$ such that

- $\mathcal{C}$ has enough projectives and injectives,
- $\{\text{projectives in } \mathcal{C}\} = \{\text{injectives in } \mathcal{C}\}$.

Theorem (Happel)**.** *Let $\mathcal{C}$ be a Frobenius exact category and $\underline{\mathcal{C}} = \mathcal{C}/[\{\text{projectives}\}]$. Then $\underline{\mathcal{C}}$ carries a triangulated structure such that*

- *triangles come from conflations,*
- *$M[1] = \Omega^-(M)$ (a "cosyzygy") for each $M \in \mathcal{C}$.*

Observation. *If $\mathcal{C}$ is Frobenius, $\mathrm{Ext}^1_{\mathcal{C}}(M, K) \cong \mathrm{Hom}_{\underline{\mathcal{C}}}(M, K[1])$ naturally for every $M, K \in \mathcal{C}$.*

Then Hom-orthogonal pairs and existence of approximation triangles translate into the following concepts, which in fact make sense for **any** (not necessarily Frobenius) exact category:

Cotorsion pair: A pair $(\mathcal{A}, \mathcal{B})$ of full subcategories of $\mathcal{C}$ such that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{A}^{\perp_1} = \mathcal{B}$ (orthogonality with respect to $\mathrm{Ext}^1_{\mathcal{C}}$).

Complete cotorsion pair: A cotorsion pair such that for each $M \in \mathcal{C}$ there do exist conflations

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow B' \longrightarrow A' \longrightarrow M \longrightarrow 0$$

with $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$.

3 Efficient exact categories

To construct complete cotorsion pairs, we need more assumptions on our exact categories. What we define is, in a sense, an analogue of a Grothendieck category in the world of exact categories. The concept here is a simplified and more restrictive version of that in arXiv:1005.3248.

Efficient exact category: An exact category $\mathcal{C}$ with splitting idempotents and satisfying

left exactness: Given a well-ordered continuous direct system

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots \longrightarrow X_\omega \longrightarrow X_{\omega+1} \longrightarrow \cdots$$

of inflations, a colimit X exists and the colimit morphisms

$$X_\alpha \longrightarrow X$$

are inflations again.

smallness: Each object of $\mathcal{C}$ is small with respect to well-ordered continuous chains of inflations (technical).

existence of generators: $\mathcal{C}$ has enough projectives or a generator (i.e. there is $G \in \mathcal{C}$ such that each $M \in \mathcal{C}$ admits a deflation $\coprod G \twoheadrightarrow M$).

Examples.

1. $\mathcal{C} = \mathbf{C}(\text{Mod-}R)$, conflations = all exact sequences.

   Then $\mathcal{C}$ is efficient and has enough projectives and injectives.

   $$\cdots \longrightarrow R^2 \xrightarrow{\binom{01}{00}} R^2 \xrightarrow{\binom{01}{00}} R^2 \longrightarrow \cdots$$

   is a (projective) generator of $\mathcal{C}$.

2. $\mathcal{C} = \mathbf{C}(\mathcal{G})$, $\mathcal{G}$ Grothendieck, conflations = all exact sequences.

   Then $\mathcal{C}$ is efficient and has a generator and enough injectives. In fact, $\mathcal{C}$ itself is a Grothendieck category. However, $\mathcal{C}$ need not have enough projectives.

3. $\mathcal{C} = \mathbf{C}(\mathcal{G})$, conflations = componentwise split exact sequences.

   $\mathcal{C}$ is again efficient and has enough projectives and injectives.

   Beware: For $\mathcal{C} = \mathbf{C}(Ab)$, no single object $G \in \mathcal{C}$ is a generator!

Before stating the main result, we need to formalize transfinite extensions in an efficient exact category $\mathcal{C}$. Let $\mathcal{S} \subseteq \mathcal{C}$ be a class of objects. We define:

$\mathcal{S}$-filtration: A continuous well-ordered chain of inflations

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots \longrightarrow X_\omega \longrightarrow X_{\omega+1} \longrightarrow \cdots$$

such that $\operatorname{Coker}(X_\alpha \longrightarrow X_{\alpha+1}) \in \mathcal{S}$ for each α.

$\mathcal{S}$-filtered object: A colimit of an $\mathcal{S}$-filtration.

Denoting the class of all $\mathcal{S}$-filtered objects in $\mathcal{C}$ by Filt-$\mathcal{S}$, we can formulate the main result. Its proof is based on Quillen's small object argument.

Theorem (Saorín, Š.). *Let $\mathcal{C}$ be an efficient exact category and $\mathcal{S} \subseteq \mathcal{C}$ a set (not a proper class!) of objects. Then:*

1. *If $\mathcal{S}$ contains a generator of $\mathcal{C}$, $({}^{\perp_1}(\mathcal{S}^{\perp_1}), \mathcal{S}^{\perp_1})$ is a complete cotorsion pair. Moreover, ${}^{\perp_1}(\mathcal{S}^{\perp_1})$ consists precisely of summands of $\mathcal{S}$-filtered objects.*

2. *If $\mathcal{C}$ has enough projectives, $({}^{\perp_1}(\mathcal{S}^{\perp_1}), \mathcal{S}^{\perp_1})$ is a complete cotorsion pair. Moreover, ${}^{\perp_1}(\mathcal{S}^{\perp_1})$ consists precisely of summands of objects E appearing in a conflation*

$$0 \longrightarrow P \longrightarrow E \longrightarrow F \longrightarrow 0$$

   with P projective and $F \in$ Filt-$\mathcal{S}$.

4 Semiorthogonal decompositions and filtrations

We can use the results to compare

- approximation theory for infinitely generated modules (e.g. a book by Göbel and Trlifaj),
- theory for well-generated triangulated categories (a book by Neeman).

Theorem (Saorín, Š.). *Let $\mathcal{C}$ be an accessible efficient Frobenius exact category and let $\mathcal{S} \subseteq \underline{\mathcal{C}}$ be a set such that $\mathcal{S} = \mathcal{S}[1]$. Then:*

1. $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ *is a semiorthogonal decomposition of* $\underline{\mathcal{C}}$ *and* ${}^{\perp}(\mathcal{S}^{\perp})$ *is a well generated triangulated category (*$\underline{\mathcal{C}}$ *itself need not be!),*
2. ${}^{\perp}(\mathcal{S}^{\perp})$ = *the closure of* $\mathcal{S}$ *under coproducts and triangle completions,*
   = *the essential image of* Filt-$\mathcal{S}$ *under* $\mathcal{C} \twoheadrightarrow \underline{\mathcal{C}}$.

A short comment on the unexplained terminology:

Accessible category: A technical condition; $\mathbf{C}(\text{Mod-}R)$ or $\mathbf{C}(\mathcal{G})$, $\mathcal{G}$ a Grothendieck category, are always accessible.

Well-generated triangulated category: A concept defined by Neeman in his book, there is a very satisfactory theory for Bousfield localizations of such categories.

Example. Let $\mathcal{C} = \mathbf{C}(\text{Mod-}R)$, conflations = componentwise split exact sequences. Then $\mathcal{C}$ is accessible efficient Frobenius.

Let $\mathcal{S}$ be a representative set of all bounded below complexes of free modules of finite rank:

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow R^{r_n} \longrightarrow R^{r_{n+1}} \longrightarrow R^{r_{n+2}} \longrightarrow \cdots$$

Then Filt-$\mathcal{S} = \mathbf{C}(\text{Free-}R)$, so $\mathbf{K}(\text{Free-}R)$ is well-generated. Eilenberg's swindle shows that

$$\mathbf{K}(\text{Free-}R) \xrightarrow{\sim} \mathbf{K}(\text{Proj-}R),$$

and we recover results of Jørgensen and Neeman, that $\mathbf{K}(\text{Proj-}R)$ is well generated.

5 The mock homotopy category of projectives

A more elaborate example due to Neeman and Murfet: Let $\mathbb{X}$ be a nice enough scheme (quasi-compact and separated) and let

$$\mathcal{F} = \{\text{flat quasi-coherent sheaves}\}.$$

Further, let $\tilde{\mathcal{F}} \subseteq \mathbf{C}(\mathcal{F})$ be the subclass of all acyclic complexes with flat cycle objects. Then Murfet defines

Mock homotopy category of projectives: the Verdier quotient $\mathbf{K}(\mathcal{F})/\tilde{\mathcal{F}}$.

Remarks.

1. Terminology: If $\mathbb{X}$ is affine, then $\mathbf{K}(\mathcal{F})/\tilde{\mathcal{F}} \xrightarrow{\sim} \mathbf{K}(\text{Proj-}\mathbb{X})$ (Neeman).
2. Motivation: $\mathbf{K}(\mathcal{F})/\tilde{\mathcal{F}}$ is compactly generated and the category of compact objects is equivalent to $\mathbf{D}^b(\mathfrak{coh}\mathbb{X})^{\text{op}}$ (Murfet).

Question. *Is there a semiorthogonal decomposition* $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}^{\perp})$ *of* $\mathbf{K}(\mathcal{F})$*?*

The answer is YES, but:

Warning. Typically, there is **no** set $\mathcal{S} \subseteq \mathbf{K}(\mathcal{F})$ such that $\mathcal{S}^{\perp} = \tilde{\mathcal{F}}^{\perp}$ in $\mathbf{K}(\mathcal{F})$!

Solution: We take another exact structure on $\mathbf{C}(\mathcal{F})$. Not the Frobenius one, but the one with:

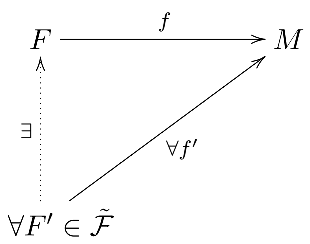
$$\text{conflations} = \textbf{all} \text{ s.e.s. of complexes of flat quasi-coherent sheaves.}$$

For this exact structure, there is a generating set $\mathcal{S}' \subseteq \mathbf{C}(\mathcal{F})$ such that $\mathcal{F} = \text{Filt-}\mathcal{S}'$, and so $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}^{\perp_1})$ is a complete cotorsion pair.

Warning. We have changed the exact structure, so typically $\tilde{\mathcal{F}}^{\perp} \neq \tilde{\mathcal{F}}^{\perp_1}$!

But we know, using the approximation conflations for $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}^{\perp_1})$, that $\tilde{\mathcal{F}}$ is contravariantly finite in $\mathbf{C}(\mathcal{F})$, and so also in $\mathbf{K}(\mathcal{F})$. Here:

Contravariant finiteness: $\forall M \in \mathbf{C}(\mathcal{F})$, there is $f\colon F \longrightarrow M$ with $F \in \tilde{\mathcal{F}}$ and such that



Now, we just apply the following lemma.

Lemma (Neeman; Keller, Vossieck)**.** *Let $\mathcal{T}$ be a triangulated category with splitting idempotents. Let $\mathcal{X} \subseteq \mathcal{T}$ be a full* **suspended** *subcategory, i.e. closed under extentions, summands and $\mathcal{X}[1] \subseteq \mathcal{X}$. Then*

$$\mathcal{X} \text{ is contravariantly finite in } \mathcal{T} \quad \Longleftrightarrow \quad (\mathcal{X}, \mathcal{X}^{\perp}) \text{ is a t-structure.}$$