Algebraic t-structures for piecewise hereditary algebras

§1) Recollements, induction and restriction
§2) Algebraic t-structures (looking at "simples")
§3) Silting objects and hearts (looking at "projectives")

§1) Recollements, induction and restriction

$\mathcal{X}, \mathcal{Y}, \mathcal{D}$ triangulated categories
Suppose we have a recollement

$$\mathcal{Y} \rightleftarrows \mathcal{D} \rightleftarrows \mathcal{X}$$

and t-structures $(\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0}) = T_{\mathcal{Y}}$ and $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0}) = T_{\mathcal{X}}$ on $\mathcal{Y}$ and $\mathcal{X}$ resp.

Thm (BBD, 1982)

- $D^{\leq 0} = \{X \in D : j^* X \in \mathcal{X}^{\leq 0}, i^* X \in \mathcal{Y}^{\leq 0}\}$
  $D^{\geq 0} = \{X \in D : j^* X \in \mathcal{X}^{\geq 0}, i^! X \in \mathcal{Y}^{\geq 0}\}$ forms a t-structure in D. (denote it by $\mathrm{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})$)
- A t-structure $(D^{\leq 0}, D^{\geq 0})$ is induced from a fixed recollement if and only if $j_! j^* (D^{\leq 0}) \subseteq D^{\leq 0}$ (i.e., $j_! j^*$ is right t-exact).

Rmks: • if $j_! j^*$ is right t-exact, then define $\mathrm{Res}((D^{\leq 0}, D^{\geq 0})) = (T_{\mathcal{X}}, T_{\mathcal{Y}})$
where $T_{\mathcal{X}} = (j^* D^{\leq 0}, j^* D^{\geq 0})$ and $T_{\mathcal{Y}} = (i^* D^{\leq 0}, i^! D^{\geq 0})$.

Lemma: • $T_{\mathcal{X}}$ and $T_{\mathcal{Y}}$ are nondegenerate iff $\mathrm{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})$ is nondeg.
• $T_{\mathcal{X}}$ and $T_{\mathcal{Y}}$ are bounded iff $\mathrm{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})$ is bdd.
• The hearts $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ are length categories iff the
(fin-tely many simples?) heart of $\mathrm{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})$ is a length category

Questions: if $\mathcal{D} = D^b(A)$, A piecewise hereditary (i.e., $D^b(A) \cong D^b(\mathcal{H})$ where $\mathcal{H}$ is hereditary abelian category) can we:

1) Describe which t-structures are induced from derived categories? A: § 2)

2) Describe the hearts of such t-structures. A: § 3).

§ 2) Algebraic t-structures

Def.: A f. dim. over $\Bbbk$ (field).
A family $X_1, \dots, X_n$ of objects in $D^b(\mathrm{mod}\, A) =: D^b(A)$ is a family of simple-minded objects if the following hold:

(1) $\mathrm{Hom}(X_i, X_j[m]) = 0 \quad \forall m < 0$

(2) $\mathrm{Hom}(X_i, X_j) = \delta_{ij} \Bbbk$, δ_{ij} kronecker delta

(3) $\mathrm{tria}(X_1 \oplus \dots \oplus X_n) = D^b(A)$.

Thm (Keller-Nicolas / König-Yang): There is a bijection between

$$T_{alg} = \left\{ \begin{matrix} \text{bdd t-structures in } D^b(A) \end{matrix} \;\middle|\; \begin{matrix} \text{heart is length category} \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{families of simple-} \\ \text{minded objects} \end{matrix} \right\} \Big/ \sim$$

$$(D^{\le 0}, D^{\ge 0}) \longmapsto \{\text{simples in } D^{\le 0} \cap D^{\ge 0}\}$$

Moreover, the heart of a bdd t-structure is a length category if and only if it is equivalent to $\mathrm{mod}\, \Gamma$ for some f.d Γ over k.

Def: An algebraic t-structure is an element of T_{alg}.

Specify to $D^b(A)$ where A is piecewise hereditary. Let $\mathrm{mod}\, \Gamma$ be the heart of an algebraic t-structure.

Fact 1: Γ is a finite dimensional directed algebra (follows from Happel's work on pw hereditary algebras). In particular each X_i is exceptional)

simple-minded.

Why is this important? We want to find a recollement by derived categories compatible with the t-structure ---

(i.e., such that $j_!\, j^*$ is right t-exact).

<u>Prop.</u> (König, Liu, Hügel): A pw hereditary over $\Bbbk$. For any indecomposable exceptional $X \in D^b(A)$, there is a recollement
$$D^b(B) \rightleftarrows D^b(A) \rightleftarrows D^b(C)$$
where B and C are pw hereditary (in fact C is a division ring over $\Bbbk$, more precisely $C = \mathrm{End}(X)$)

<u>Theorem</u> (Liu, — '2011): A pw hereditary algebra over $\Bbbk$.

A t-structure $(D^{\leq 0}, D^{\geq 0})$ is algebraic in $D^b(A)$ if and only if it is induced by a recollement of derived categories of modules.

<u>idea of proof</u>: one direction is more or less easy from the previous section.

The main problem is: given algebraic t-structure find compatible recollement.

<u>Trick</u>: Consider the simple-minded objects. Algebra Γ is directed, hence at least one of them is projective.

Among the set of indecomposable simple projective Γ-modules choose X_i such that $X_i \in \mathcal{H}[\ell]$ where ℓ is maximal (recall that $D^b(\mathcal{H}) = \bigoplus \mathcal{H}[\ell]$ since $\mathcal{H}$ is hereditary).

<u>Then</u>: 1) We have a recollement (since X_i is exceptional!)
$$\mathcal{Y} \rightleftarrows D^b(A) \rightleftarrows D^b(\mathrm{End}(X_i))$$

2) $\mathcal{Y} = D^b(B)$ for a pw hereditary algebra B by choice of X_i.

3) $j_!\, j^*$ is right t-exact (easy.) ☒

E.g.: $D^b(\cdot \rightrightarrows \cdot) \cong D^b(\mathrm{coh}\,\mathbb{P}^1)$

$\mathrm{coh}\,\mathbb{P}^1$ is a heart of a t-structure which is not algebraic and therefore not induced by recollements of derived categories of modules.

derived categories of modules?

Cor.: if additionally A is of f^te rep type, then every bold t-structure is algebraic and thus induced from derived categories of modules

§3) Silting objects and hearts

Characterise heart Γ.

Def.: A silting object $S \in D^b(A)$ is an object satisfying: (1) $\mathrm{Hom}(S, S[i]) = 0 \ \forall i > 0$
(2) S generates $D^b(A)$ as a triangulated category.

Thm (Keller, Vossieck):

If Δ is a Dynkin quiver, there is a bijection between the silting objects and ______ t-structures in $D^b(A)$.

Def: $X \in D^{\leq 0}$ is Ext-projective $\underset{\text{def}}{\iff}$ $\mathrm{Hom}(X, Y[1]) = 0 \ \forall Y \in D^{\leq 0}$.

Thm (Assem / Souto Salorio / Trepode): Let A be pw hereditary algebra over $\Bbbk$. The following statements hold in $D^b(A)$.

- There is a fully faithful correspondence

holds in full generality!

$$H^0 : \left\{ \begin{array}{c} \text{Ext-projectives} \\ \text{in } D^{\leq 0} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{projectives} \\ \text{in } D^{\leq 0} \cap D^{\geq 0} \end{array} \right\}$$

- $\underset{\substack{X_i \text{ indec.} \\ \text{Ext-proj.} \\ \text{in } D^{\leq 0}}}{\bigoplus} X_i$ is silting $\iff$ $\#$Ext-projs $= \mathrm{rk}\, K_0(D^b(A))$

- smallest suspended subcategory containing X is an aisle.

Thm: Let A be piecewise hereditary. There is a bijection

$$\left\{\begin{array}{c}\text{basic silting}\\ \text{objects in}\\ D^b(A)\end{array}\right\} \overset{\cong}{\longleftrightarrow} \left\{\begin{array}{c}\text{algebraic } t\text{-structures}\\ \text{in } D^b(A)\end{array}\right\}$$

$$X \longmapsto (\mathcal{U}_X, \mathcal{U}_X^{\perp}[1])$$

$$\bigoplus_{i=1}^{\mathrm{rk}\, A} X_i \longleftarrow (D^{\leq 0}, D^{\geq 0})$$

Where $\{X_1, \ldots, X_{\mathrm{rk}\,A}\}$ are the indecomposable Ext-projectives in $D^{\leq 0}$ (pairwise non-isomorphic).

Moreover, $\mathcal{U}_X \cap \mathcal{U}_X^{\perp}[1] \cong \mathrm{mod}\, \mathrm{End}(X)$.

"Proof": <u>key observation</u>:

$$\#\left\{\begin{array}{c}\text{indec. Ext-proj. in } D^{\leq 0}\\ \text{up to iso}\end{array}\right\} = \mathrm{rk}(K_0(A))$$

$$\Longleftrightarrow (D^{\leq 0}, D^{\geq 0}) \text{ is algebraic.}$$

Then it is easy to see that

$$H^0 : \left\{\begin{array}{c}\text{indec. Ext-proj}\\ \text{in } D^{\leq 0}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{projectives}\\ \text{in } D^{\leq 0} \cap D^{\geq 0}\end{array}\right\}$$

is a bijection and thus

$$\mathrm{End}(\oplus X_i) = \mathrm{End}(\oplus P_i)$$

and $\oplus P_i$ is a progenerator in the heart (since the heart is module category).

Rmk: $X \in \mathcal{U}_X \cap \mathcal{U}_X^{\perp}[1] \Leftrightarrow X$ is tilting $\Leftrightarrow D^b(A) \cong D^b(\mathrm{mod}\, \mathrm{End}(X))$

"Proof" of key observation:

$\Rightarrow$ easy

$\Leftarrow$ this is equivalent to prove that the exceptional collection formed by the indec. Ext-projectives generates the derived category. It follows from the following proposition:

Prop.: $H^0 : D^b(A) \longrightarrow \mathcal{A} := D^{\leq 0} \cap D^{\geq 0}$

Choose $\phi : \mathcal{A} \longrightarrow \text{mod}\,\Gamma$ given by basic progenerator (direct sum of indec. projectives). Then

$$\overline{H}^0 : D^b(A) \longrightarrow \mathcal{A} \xrightarrow{\phi} \text{mod}\,\Gamma \xrightarrow{u} \text{mod}\,k$$

is representable and is represented by X.

We can now define glueing of silting objects just using glueing of t-structures and the theorem stating that the glued t-structure of two algebraic t-structures is still algebraic. Moreover the same theorem shows that all silting objects are glued in this way!