J. Wildeshaus, "f-categories and their applications"

§1 Statement of the main result

§2 f-categories à la Beilinson

§3 Proof of the main result

arxiv.org/abs/0810.1674

§1

Data: $(\mathcal{C}, t)$: Δ cat., with a t-str.,
$\mathcal{C}^0$: its heart.

Question: Can one extend

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, hook] \arrow[d, equal] & D^b(\mathcal{C}^0) \arrow[d, dashed, "\Delta(?)\ ?"] \\ \mathcal{C}^0 \arrow[r, hook] & \mathcal{C} \end{tikzcd}$$

History:
- Beilinson–Bernstein–Deligne (1982):
  OK if $\mathcal{C} \hookrightarrow D^+(\mathcal{A})$,
  $\mathcal{A}$ Ab. cat. with suff^{ly} many injective objects

{ uses the derived filtered cat. $D^+F(\mathcal{A})$

- Keller–Vossieck (1987)

{ simplification of this approach

- then: two generalizations of the BBD-criterion:

today: Beilinson (1987):
OK if $\mathcal{C}$ admits an f-category over $\mathcal{C}$

Keller (1991):
OK if $\mathcal{C}$ is the base of a "tower equivalence" of Δ categories

{ Thus, in both cases, need to "enrich" $\mathcal{C}$...

Theorem: (W.) Any ~~[illegible]~~ fully faithful triangulated (but not nec.ly t-exact) immersion

$$\mathcal{C} \hookrightarrow D(\mathcal{A})$$

of $\mathcal{C}$ into the (unbounded) der. cat. of an exact cat. $\mathcal{A}$ induces a triang. functor

$$\mathrm{real}: D^b(\mathcal{C}^{\heartsuit}) \longrightarrow \mathcal{C}$$

inducing the identity on $\mathcal{C}^{\heartsuit}$ (hence t-exact).

Note: The triangulated category of effective motives à la Voevodsky embeds into

$$D^-(\text{some Ab. cat.}) \qquad !$$

§2 (Beilinson, 1987.)

Definition: (a) An *f-category* is a Δ cat. $\mathcal{CF}$, together with

$$f = (\mathcal{CF}(\leq 0), \mathcal{CF}(\geq 0), s, \iota) :$$

$\mathcal{CF}(\leq 0), \mathcal{CF}(\geq 0)$: full sub-cat., Δ, strict,

$$\begin{tikzcd} \mathcal{CF} \arrow[r, "\sim"] & \mathcal{CF} \arrow[l, bend left, "s^{-1}"] \end{tikzcd}$$

$s : \mathcal{CF} \xrightarrow{\sim} \mathcal{CF}$ auto-equiv., Δ,

$\iota : \mathrm{id} \to s$ transf. of func.,

such that, putting $\mathcal{CF}(\leq n) := s^n\mathcal{CF}(\leq 0)$, $\mathcal{CF}(\geq n) = s^n\mathcal{CF}(\geq 0)$

(1) $\bigcup_{n\in\mathbb{Z}} \mathcal{CF}(\leq n) = \mathcal{CF} = \bigcup_{n\in\mathbb{Z}} \mathcal{CF}(\geq n)$,

$$\mathcal{CF}(\leq 0) \subset \mathcal{CF}(\leq 1), \qquad \mathcal{CF}(\geq 0) \supset \mathcal{CF}(\geq 1),$$

(2) $\forall X \in \mathcal{CF}$, $\iota_X = s(\iota_{s^{-1}X}) : X \longrightarrow sX$,

(3) $\forall X \in \mathcal{CF}(\geq 1)$, ~~[illegible]~~ $Y \in \mathcal{CF}(\leq 0)$,

$$\mathrm{Hom}_{\mathcal{CF}}(X, Y) = 0,$$

$$\mathrm{Hom}_{\mathcal{CF}}(Y, s^{-1}X) \xrightarrow[\sim]{\iota_*} \mathrm{Hom}_{\mathcal{CF}}(Y, X) \xleftarrow[\sim]{\iota^*} \mathrm{Hom}_{\mathcal{CF}}(sY, X),$$

(4) $\forall X \in \mathcal{C}F \ \exists$ ex. Δ

$$\left.\begin{array}{l} A \longrightarrow X \longrightarrow B \longrightarrow A[1], \\ A \in \mathcal{C}F(\geq 1), \ B \in \mathcal{C}F(\leq 0) \end{array}\right\} \underset{(8)}{\Longrightarrow} \text{this } \Delta \text{ is unique up to unique iso.}$$

(b) Let $\mathcal{C}$ be an Δ cat. An __f-cat. over $\mathcal{C}$__ is a pair $(\mathcal{C}F, g)$:

$\mathcal{C}F$: an f-cat.,

$g: \mathcal{C} \underset{\Delta}{\xrightarrow{\sim}} \mathcal{C}F(\leq 0) \cap \mathcal{C}F(\geq 0)$.

__Example:__ $\mathcal{A}$ abel. cat.,

$\mathcal{C} = D^{\cdot}(\mathcal{A})$,

$\mathcal{C}F = D^{\cdot}F(\mathcal{A})$, the __derived filtered category__ of $\mathcal{A}$

$\{$ of complexes with a __finite__ __decreasing__ filtr.

__Constructions:__ $\{$ generalising the "evident" ones for $D^{\cdot}F(\mathcal{A})$

$$(a)\ n \in \mathbb{Z}: \left.\begin{array}{ll} \sigma_{\leq n}: \mathcal{C}F \longrightarrow \mathcal{C}F(\leq n) & \text{left} \\ \sigma_{\geq n}: \mathcal{C}F \longrightarrow \mathcal{C}F(\geq n) & \text{right} \end{array}\right\} \text{adj. to the incl.}$$

$\rightarrow$ ex. Δ $\quad \sigma_{\geq n+1} \longrightarrow \mathrm{id} \longrightarrow \sigma_{\leq n} \longrightarrow \sigma_{\geq n+1}[1]$,

$gr_f^n: \mathcal{C}F \longrightarrow \mathcal{C} \ ; \ g^{-1} s^{-n} \sigma_{\leq n} \sigma_{\geq n} = g^{-1} s^{-n} \sigma_{\geq n} \sigma_{\leq n}$.

(b) $\exists\, \omega: \mathcal{C}F \longrightarrow \mathcal{C}$ generalising the forgetful functor.

__Definition:__ Let $(\mathcal{C}F, g)$ be an f-cat over $\mathcal{C}$, and $(\mathcal{C}, t)$ a t-str. A t-str. $(\mathcal{C}F, t)$ is said to be __compatible__ with $(\mathcal{C}, t)$ if

(1) $g: \mathcal{C} \longrightarrow \mathcal{C}F$ is t-exact,

(2) $s(\mathcal{C}F^{t \leq 0}) = \mathcal{C}F^{t \leq -1}$.

Theorem: (Beilinson)

$(\mathcal{C}F, \gamma)$ an f-cat. over $\mathcal{C}$, $(\mathcal{C}, t)$ a t-str.

(a) $\exists!$ t-str. $(\mathcal{C}F, t)$ comp. with $(\mathcal{C}, t)$.

(b) $\mathcal{C}^\circ$, $\mathcal{C}F^\circ$ the hearts $\subset \mathcal{C}$, $\mathcal{C}F$,

$\mathcal{H} := \tau^{t\le 0} \circ \tau^{t\ge 0} : \mathcal{C} \longrightarrow \mathcal{C}^\circ$. Define

$\mathcal{H}F : \mathcal{C}F \longrightarrow C^b(\mathcal{C}^\circ)$ as follows:

$X \longmapsto (\mathcal{H}F(X)^n, d^n)_{n\in\mathbb{Z}}$, where

$\mathcal{H}F(X)^n := \mathcal{H}(gr_f^n X)$;

d^n induced by

$\mathcal{H}(gr_f^{n+1} X \longrightarrow \sigma_{\le n+1}\sigma_n X \longrightarrow gr_f^n X \xrightarrow{+1})$.

Then $\mathcal{H}F|_{\mathcal{C}F^\circ} : \mathcal{C}F^\circ \longrightarrow C^b(\mathcal{C}^\circ)$ is an equiv. of cat.,

and $(\mathcal{H}F|_{\mathcal{C}F^\circ})^{-1} \circ \mathcal{H}F : \mathcal{C}F \longrightarrow \mathcal{C}F^\circ$ is the ~~f~~ natural cohomol. functor "$\tau^{t\ge 0} \circ \tau^{t\le 0}$" on $\mathcal{C}F$.

Main Construction: In the above situation,

$\widetilde{real} : C^b(\mathcal{C}^\circ) \xrightarrow{(\mathcal{H}F|_{\mathcal{C}F^\circ})^{-1}} \mathcal{C}F^\circ \hookrightarrow \mathcal{C}F \xrightarrow{\omega} \mathcal{C}$.

one verifies: $\widetilde{real}$ factors through $D^b(\mathcal{C}^\circ)$,

and induces id on $\mathcal{C}^\circ$.

§3

$(\mathcal{C}, t)$, $\mathcal{C}^\circ$, $\mathcal{C} \hookrightarrow D(\mathcal{A})$ as in §1.

§2 $\Longrightarrow$ have to deduce from

$\mathcal{C} \longrightarrow D(\mathcal{A})$

an f-cat. $\mathcal{C}F$ over $\mathcal{C}$.

~~above~~ Example from §2: can^l. f-cat. $DF(\mathcal{A})$ over $D(\mathcal{A})$.

Construction: $\mathcal{C}F := \{X \in DF(\mathcal{A}),\ gr_f^n X \in \mathcal{C} \subset D(\mathcal{A})\ \forall n \in \mathbb{Z}\}$.

one verifies: $\mathcal{C}F$ is an f-cat. over $\mathcal{C}$.

{re- and reprint!