PROPOSITION.- **Let $\mathcal{D}$ be a triangulated category with small coproducts and $\mathcal{D}'$ be an arbitrary triangulated subcategory. If for each aisle $\mathcal{U}$ of $\mathcal{D}'$, $Susp_{\mathcal{D}}(\mathcal{U})$ is an aisle of $\mathcal{D}$, then each t-estructure on $\mathcal{D}'$ is the restriction of a t-structure on $\mathcal{D}$. Moreover, in such a case the assignment $\mathcal{U} \rightsquigarrow Susp(\mathcal{U})$ gives a bijection between the aisles of $\mathcal{D}'$ and the aisles of $\mathcal{D}$ which restrict to $\mathcal{D}'$.**

COROLLARY.- **Under the hypotheses above, suppose in addition that $\mathcal{D}$ is compactly generated. If for each aisle $\mathcal{U}$ of $\mathcal{D}'$, one has that $Susp_{\mathcal{D}}(\mathcal{U}) = Susp_{\mathcal{D}}(\mathcal{S})$ for some set of compact objects $\mathcal{S}$, then each t-structure on $\mathcal{D}'$ is the restriction of a t-structure on $\mathcal{D}$.**

EXAMPLES.-

1. **If $\mathcal{D}$ is a compactly generated triangulated category, then each t-stucture on $\mathcal{D}^c$ is the restriction of a t-structure on $\mathcal{D}$. Moreover, we have a bijection between t-structures on $\mathcal{D}^c$ and compactly generated t-structures on $\mathcal{D}$.**
2. **If R is a commutative Noetherian ring, then each t-structure on $D^b_{fg}(R)$ is the restriction of a compactly generated t-structure on $D(R)$**

DEFINITION.- A map $\phi : \mathbf{Z} \longrightarrow \mathcal{P}(Spec(R))$ is a *filtration by supports* if $\phi(n) \supseteq \phi(n+1)$ and $\phi(n)$ is closed under specialization, for all $n \in \mathbf{Z}$.

THEOREM (A-J-S).- Let R be a commutative Noetherian ring. Then each aisle in $D(R)$ generated by complexes of $D^b_{fg}(R)$ is compactly generated. Moreover, if for each filtration by supports $\phi : \mathbf{Z} \longrightarrow \mathcal{P}(Spec(R))$ we put

$$\mathcal{U}_\phi := \{X \in D(R) : \\ Supp(H^i(X)) \subseteq \phi(i), \text{ for all } i \in \mathbf{Z}\}.$$

then

$$\mathcal{U}_\phi = Susp_{D(R)}(\frac{R}{\mathbf{p}}[-i] : i \in \mathbf{Z} \text{ and } \mathbf{p} \in \phi(i)).$$

and the assignment $\phi \rightsquigarrow \mathcal{U}_\phi$ establishes a bijection between filtrations by supports on $Spec(R)$ and compactly generated t-structures on $D(R)$.

QUESTION 1.- Which are the compactly generated t-structures of $D(R)$ which restrict to t-structures on $D^b_{fg}(R)$?.

Equivalently:

QUESTION 2.- Which are the filtrations by supports ϕ on $Spec(R)$ such that $\mathcal{U}_\phi \cap D^b_{fg}(R)$ is an aisle of $D^b_{fg}(R)$?

THEOREM (A-J-S).- Let R be a commutative Noetherian ring and $\phi : \mathbf{Z} \longrightarrow \mathcal{P}(Spec(R))$ be a filtration by supports. Consider the following assertions:

1. **$\mathcal{U}_\phi \cap D^b_{fg}(R)$ is an aisle of $D^b_{fg}(R)$**
2. **If $\mathbf{p} \subsetneq \mathbf{q}$ are prime ideals, with $\mathbf{p}$ maximal under $\mathbf{q}$, and $\mathbf{q} \in \phi(j)$ then $\mathbf{p} \in \phi(i)$ (we will say in this case that ϕ satisfies the *weak Cousin condition* (wCc)).**

Then $1) \Longrightarrow 2)$ and, in case R has a dualizing complex, the converse is also true.

KEY IDEAS FOR THE PROOF OF $2) \Longrightarrow 1)$**:**

- W.l.o.g. we assume that $Spec(R)$ **is connected and fix a dualizing complex** D**. Then we know:**

a) $D \in D^b_{fg}(R)$ **and** D **is quasi-isomorphic to a bounded complex of injectives**

b) There is a codimension function $d : Spec(R) \longrightarrow \mathbf{Z}$ **defined by the rule:**

$$d(\mathbf{p}) = j \Longleftrightarrow Hom_{D(R_{\mathbf{p}})}(k(p), D_{\mathbf{p}}[j]) \neq 0.$$

c) R **has finite Krull dimension.**

d) $RHom_R(-, D) : D^b_{fg}(R) \xrightarrow{\cong} D^b_{fg}(R)$ **is a duality.**

- LEMMA.- If $H := RHom_R(-, D)$ then the assignement $\mathcal{U} \rightsquigarrow^{\perp} H(\mathcal{U})$ gives a bijection inverse of itself between the set of *closed pre-aisles* (resp. aisles) in $D^b_{fg}(R)$ and itself. In particular, it induces a bijection between the corresponding sets of filtrations by supports on $Spec(R)$ and themselves.

- EXAMPLES.-

a) **If ϕ is the filtration by supports on $Spec(R)$ correspoding to the canonical t-structure, then its image by the above bijection is he filtration ϕ_{CM} given by**

$$\phi_{CM}(i) = \{\mathbf{p} \in Spec(R) : d(\mathbf{p}) > i\}$$

b) **If $Z \subseteq Spec(R)$ is closed under specialization and $n \in \mathbf{Z}$ is an integer, then the filtration by supports $\phi_{Z,n}$ given by $\phi_{Z,n}(i) = Z$, for $i \leq n$, and $\phi_{Z,n}(i) = \emptyset$, for $i > n$, has as its image by the above map the filtration by supports $\xi = \xi_{Z,n}$ given by**

$$\xi(k) = \{\mathbf{q} \in Spec(R) : \\ V(\mathbf{q}) \cap Z \subseteq \phi_{CM}(k+n)\}.$$

OBSERVATION.- Denote by $\Gamma_Z : R-Mod \longrightarrow R-Mod$ **the torsion radical associated to** Z **and by** $R\Gamma_Z : D(R) \longrightarrow D(R)$ **is right derived functor. The left truncation functor associated to the aisle** $\mathcal{U}_{\phi_{Z,n}}$ **is precisely** $\tau^{\leq n}R\Gamma_Z : D(R) \longrightarrow D(R)$**.**

If $M \in D^b_{fg}(R)$**, then we have a chain of implications:**

$$R\Gamma_Z(M) \in D^{>n}(R) \Longleftrightarrow \tau^{\leq n}R\Gamma_Z(M) = 0 \Longleftrightarrow M \in \mathcal{U}^{\perp}_{\phi_{Z,n}} \Longleftrightarrow RHom_R(M,D) \in \mathcal{U}_\xi \Longrightarrow Supp(Hom_{D(R)}(M,D[k])) \subseteq \xi(k), \text{ for all } k \in \mathbf{Z} \Longleftrightarrow Z \cap Supp(Hom_{D(R)}(M,D[k])) \subseteq \phi_{CM}(k+n), \text{ for all } i \in \mathbf{Z}.$$

LEMMA.- With the notation above, let M **be in** $D^b_{fg}(R)$**. The following assertions are equivalent:**

1. $\tau^{\leq n}R\Gamma_Z(M)$ **is** $D^b_{fg}(R)$
2. $Supp(Hom_{D(R)}(M,D[k])) \subseteq Z \cup \xi(k)$**, for all** $i \in \mathbf{Z}$**.**

LEMMA.- (Always assuming that R has a dualizing complex) if $\phi : \mathbf{Z} \longrightarrow \mathcal{P}(Spec(R))$ is a filtration satisfying wCc, then it is of the form

$$... = Spec(R) = \phi(r) \supsetneq \phi(r+1) \supseteq \phi(r+2) ... \supseteq \phi(n) \supsetneq \phi(n+1) = \emptyset = ...,$$

for uniquely determined integers $r < n$

If ϕ, r and n are as in the above lemma and $n - k + 1 > 1$, then the filtration by supports

$$\phi'\text{:} \qquad ... = Spec(R) = \phi(r) \supsetneq \phi(r+1) \supseteq \phi(r+2) ... \supseteq \phi(n-1) \supsetneq \phi'(n) = \emptyset = ...$$

also satisfies the wCc. This suggests to apply induction on the length $n - r + 1$, the case of length 1 being trivial.

The key point is then

LEMMA.- With the notation above, if we put $Z = \phi(n)$, then we have triangles in $D(R)$

$$\tau^{\leq}_{\phi'}X \longrightarrow \tau^{\leq}_{\phi}X \longrightarrow \tau^{\leq n}R\Gamma_Z(\tau^{>}_{\phi'}X) \stackrel{+}{\longrightarrow}$$
$$\tau^{\leq n}R\Gamma_Z(\tau^{>}_{\phi'}X) \longrightarrow \tau^{>}_{\phi'}X \longrightarrow \tau^{>}_{\phi}X \stackrel{+}{\longrightarrow}$$

COROLLARY.- If $\mathcal{U}_{\phi'} \cap D^b_{fg}(R)$ is an aisle of $D^b_{fg}(R)$ then $\mathcal{U}_{\phi} \cap D^b_{fg}(R)$ is an aisle of $D^b_{fg}(R)$ if, and only if, $\tau^{\leq n}R\Gamma_Z(\tau^{>}_{\phi'}X) \in D^b_{fg}(R)$, where $Z = \phi(n)$.

KEY IDEAS FOR THE PROOF OF $1) \Longrightarrow 2)$:

- We assume, by reduction to absurd, that there is an inclusion $\mathbf{p} \subsetneq \mathbf{q}$, with p maximal under q, such that $\mathbf{q} \in \phi(1)$ and $\mathbf{p} \notin \phi(0)$ (there is no loss of generality).

- We fix the ϕ-triangle

$$T \longrightarrow R/\mathbf{p} \longrightarrow Y \stackrel{+}{\longrightarrow}$$

(i.e. with $T \in \mathcal{U}_\phi$ and $Y \in \mathcal{U}_\phi^\perp$) which, by hypothesis, lives in $D^b_{fg}(R)$.

- By localizing at q, we can assume that R is local and $\mathbf{q} = \mathbf{m}$ is the maximal ideal. By the exact sequence of homology applied to the triangle, one gets:

a) $T \in D^{>0}(R)$ **and** $Y \in D^{\geq 0}(R)$

b) An exact sequence $0 \to R/\mathbf{p} \xrightarrow{j} H^0(Y) \longrightarrow H^1(T) \to 0$**, where** j **is an essential monomorphism.**

c) (As a consequence of b) $Supp(H^1(T)) \subseteq Supp(H^0(Y)) = Supp(R/\mathbf{p}) = V(\mathbf{p}) = \{\mathbf{p}, \mathbf{m}\}$**, and** $\mathbf{p} \notin Supp(H^1(T))$ **since** $\mathbf{p} \notin \phi(0)$ **(and hence** $\mathbf{p} \notin \phi(1)$**).**

- It follows that $Supp(H^1(Y)) = \{\mathbf{m}\}$**, and hence** $H^1(Y)$ **is an** R**-module of finite length and there is a** $r > 0$ **such that** $\mathbf{m}^r H^1(Y) = 0$**.**

- We check that if $A = R/\mathbf{p}$ **and** $\mathbf{n} = \mathbf{m}/\mathbf{p}$**, then the canonical maps**

$$Ext^1_A(A/\mathbf{n}^k, A) \longrightarrow Ext^1_R(A/\mathbf{n}^k, A) \longrightarrow Ext^1_R(R/\mathbf{m}^k, R)$$

are both injective.

- Using the fact that A is a local Noetherian domain of Krull dimension 1, one uses the minimal injective resolution of A to see that $\mathbf{n}^{k-1}Ext^1_A(A/\mathbf{n}^k, A) \neq 0$, and hence $\mathbf{m}^{k-1}Ext^1_R(R/\mathbf{m}^k, R/\mathbf{p}) \neq 0$, for all $k > 0$

- Using now that $R/\mathbf{m}^k[-1] \in \mathcal{U}_\phi$ (since $\mathbf{m} \in \phi(1)$)and that the truncation functor $\tau^{\leq}_{\phi} : D(R) \longrightarrow \mathcal{U}_\phi$ is right adjoint to the inclusion, we get that

$$Ext^1_R(R/\mathbf{m}^k, R/\mathbf{p}) \cong$$
$$Hom_{D(R)}(R/\mathbf{m}^k[-1], R/\mathbf{p}) \cong$$
$$Hom_{D(R)}(R/\mathbf{m}^k[-1], T) \cong$$
$$Hom_R(R/\mathbf{m}^k, H^1(T)),$$

the last isomorphism due to the fact that $T \in D^{>0}(R)$**.**

- Finally, from the equality $\mathbf{m}^r H^1(T) = 0$ we get that $\mathbf{m}^r Ext^1_R(R/\mathbf{m}^k, R/\mathbf{p}) = 0$, for all $k > 0$, which is a contradiction.