NOTES OF THE TALK "CLUSTER-HEARTS AND CLUSTER-TILTING OBJECTS"

PEDRO NICOLÁS (JOINT WITH BERNHARD KELLER)

Contents

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1. Motivation and aim

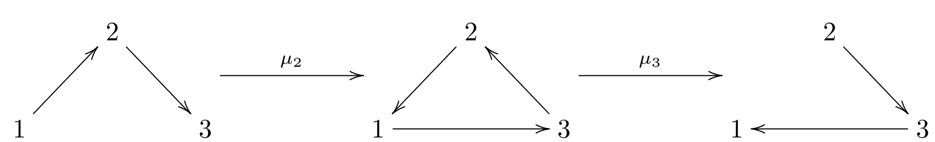
Cluster algebras were invented at around 2000 by Fomin-Zelevinsky [7]. Their main motivation was to find a combinatorial approach to Lusztig's results concerning total positivity in algebraic groups [16] and canonical bases in quantum groups [15]. However, shortly after their appearance, strong links with other areas of Mathematics were discovered: Poisson geometry, discrete dinamical systems, algebraic geometry, representation theory of finite-dimensional algebras,...

The definition of cluster algebras involves a procedure called *mutation* of a quiver at a vertex.

Definition 1.1. Let Q be a finite quiver without loops or 2-cycles (*e.g.* acyclic). The mutation of Q at the vertex r is a new quiver, $\mu_r Q$, obtained from Q following the rules:

M1) for each $i \to r \to j$ we add an arrow $i \to j$,
M2) reverse arrows incident with r,
M3) remove a maximal collection of 2-cycles.

Example 1.2.



Nowadays, a major effort is being made to understand cluster algebras by 'categorifying' them, namely, by finding nice categories encoding their combinatorics. Our aim here is to compare two categorifications of quiver mutation:

(1) via *cluster-tilting* objects (Iyama-Yoshino [10], Buan-Marsh-Reineke-Reiten-Todorov [5], Geiß-Leclerc-Schröer [8], Amiot [1], ...)
(2) via *cluster-hearts* (or *cluster collections*) (Bridgeland [3], Kontsevich-Soibelman [14], Nagao [18],...)

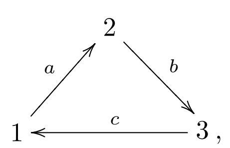
2. The setup

- k algebraically closed field,
- Q finite quiver (say $|Q_0| = n$) without loops or 2-cycles,
- $\widehat{kQ}$=completed path algebra, *i.e.* the underlying vector space is free over the set of all possible paths, and multiplications is given by concatenation of paths,
- $W \in \widehat{kQ}$ a potential = (possibly infinite) linear combination of cycles of length ≥ 3, up to cyclic equivalence ($a_1a_2 \dots a_n \sim a_2a_3 \dots a_na_1$).

Assumptions:

1) (Q, W) is a *non-degenerate*, *i.e.* no 2-cycles appear in any iterated mutation of (Q, W) (see Derksen-Weyman-Zelevinsky [6]).
2) Its Jacobian algebra $\mathcal{P}(Q, W) = \widehat{kQ}/\langle \delta_a W \ , \ a \in Q_1 \rangle$ is finite-dimensional.

Example 2.1.



$W = cba$, $\qquad \mathcal{P}(Q, W) = \widehat{kQ}/\langle ba, cb, ac \rangle$

After [9], associated to the quiver with potential (Q, W) we have the so called *complete Ginzburg dg algebra*, Γ. It is concentrated in non-positive degrees, and satisfies $H^0\Gamma = \mathcal{P}(Q, W)$. For this dg algebra we consider the following triangulated categories:

- its unbounded derived category, $\mathcal{D}\Gamma$,
- its *perfect derived category*, $\mathsf{per}\,\Gamma$, which is the smallest full triangulated subcategory of $\mathcal{D}\Gamma$ containing Γ and closed under direct summands,
- its *finite-dimensional derived category*, $\mathcal{D}_{fd}\Gamma$, which is the full subcategory of $\mathcal{D}\Gamma$ formed by those modules M such that $\Sigma_{p\in\mathbf{Z}} \dim H^pM < \infty$,

and an exact sequence of triangulated categories,

$$0 \to \mathcal{D}_{fd}\Gamma \hookrightarrow \mathsf{per}\,\Gamma \to \mathcal{C}_\Gamma \to 0.$$

$\mathcal{D}_{fd}\Gamma$ is the home of the cluster hearts (or cluster collections), and $\mathcal{C}_\Gamma$, the *generalized cluster category*, is the home of the cluster-tilting objects.

Associated to the quiver Q we have the following *braid group*,

$$\mathsf{Braid}(Q) = \langle \sigma_i \ , \ i \in Q_0 \mid \begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if there is no } i \leftrightarrow j \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if there is one } i \leftrightarrow j \end{cases} \rangle.$$

After Seidel-Thomas [23], we know that the following map is a group morphism

$$\mathsf{Braid}(Q) \to \mathsf{Auteq}(\mathcal{D}_{fd}\Gamma)\ ,\ \sigma_i \mapsto \mathsf{tw}_{S_i},$$

where

- $S_1\,,\ \dots\,,\ S_n$ is the set of simples of $H^0\Gamma$ regarded in $\mathcal{D}_{fd}\Gamma$ via the morphism $\Gamma \to H^0\Gamma$,
- tw_{S_i} is the *Seidel-Thomas twist*, defined by

$$\mathsf{RHom}(S_i, X) \otimes_k^{\mathbf{L}} S_i \xrightarrow{ev} X \to \mathsf{tw}_{S_i}(X) \xrightarrow{+}$$

3. Cluster collections

The following definition is due to Kontsevich-Soibelman [14].

Definition 3.1. A *cluster collection* is a sequence $\mathcal{S}' = (S'_1, \dots, S'_n)$ of objects of $\mathcal{D}_{fd}\Gamma$ such that

a) the S'_i are *3-spherical*, *i.e.* $H^p\,\mathsf{REnd}(S'_i) \cong H^p(\mathbb{S}^3; k) \cong \begin{cases} k \text{ if } p = 0, 3, \\ 0 \text{ else;} \end{cases}$

b) for $i \neq j$ the graded space $\bigoplus_{p\in\mathbf{Z}} \mathsf{Hom}_{\mathcal{D}\Gamma}(S'_i, \Sigma^p S'_j)$ is either zero or it is concentrated in one of the two degrees $p = 1$ or $p = 2$ only;

c) the S'_i generate $\mathcal{D}_{fd}\Gamma$.

Example 3.2. The set $\mathcal{S} = (S_1, \dots, S_n)$ of simple modules over $H^0\Gamma$ yields a cluster collection when regarded in $\mathcal{D}_{fd}\Gamma$. It is called *canonical cluster collection*. We have $\mathsf{Ext\text{-}quiver}(\mathcal{S}) = Q$

Definition 3.3. The *left Kontsevich-Soibelman(=KS)-mutation* of a cluster collection $\mathcal{S}'$ at the vertex r is the collection $\mu_{r,-}(\mathcal{S}') = (\mu_{r,-}(S'_1), \dots, \mu_{r,-}(S'_n))$ where

$$\mu_{r,-}(S'_i) = \begin{cases} \Sigma^{-1} S'_r & \text{if } i = r, \\ S_i & \text{if there is some } r \to i \text{ in } \mathsf{Ext\text{-}quiver}(\mathcal{S}'), \\ \mathsf{tw}_{S'_r}(S'_i) & \text{if } i \neq r \text{ and } r \not\to i \text{ in } \mathsf{Ext\text{-}quiver}(\mathcal{S}'). \end{cases}$$

The *right KS-mutation* of a cluster collection $\mathcal{S}'$ at the vertex r is the collection $\mu_{r,+}(\mathcal{S}') = (\mu_{r,+}(S'_1), \dots, \mu_{r,+}(S'_n))$ where

$$\mu_{r,+}(S'_i) = \begin{cases} \Sigma S'_r & \text{if } i = r, \\ S'_i & \text{if and there is some } i \to r \text{ in } \mathsf{Ext\text{-}quiver}(\mathcal{S}'), \\ \mathsf{tw}^{-1}_{S'_r}(S'_i) & \text{if } i \neq r \text{ and } i \not\to r \text{ in } \mathsf{Ext\text{-}quiver}(\mathcal{S}'). \end{cases}$$

Using ideas of Bridgeland one can prove the following:

Proposition 3.4. *For each cluster tilting collection $\mathcal{S}'$ and each $r \in \{1, \dots, n\}$:*

a) $(\mathsf{tw}_{S'_r} \circ \mu_{r,+})(\mathcal{S}') \cong \mu_{r,-}(\mathcal{S}')$.

b) $\mathsf{Ext\text{-}quiver}(\mu_{r,\varepsilon}(\mathcal{S}')) = \mu_r(\mathsf{Ext\text{-}quiver}(\mathcal{S}'))$*, for* $\varepsilon \in \{+, -\}$.

Definition 3.5. A cluster collection $\mathcal{S}'$ is *reachable* if it can be obtained from the canonical one by mutating and permutating.

4. Cluster-tilting sequences

Definition 4.1. An object $T \in \mathcal{C}_\Gamma$ is *cluster-tilting* if it is basic and $\ker \mathsf{Ext}^1(T, ?) = \mathsf{add}(T)$.

We write Q_T to refere to the Gabriel quiver of the finite-dimensional algebra $\mathsf{End}(T)$, *i.e.* the Ext-quiver of their simple modules.

Example 4.2. (Amiot [1]) $\Gamma \in \mathcal{C}_\Gamma$ cluster-tilting and $Q_\Gamma = Q$.

The following two theorems show how cluster-tilting objects help us to categorify quiver-mutation:

Theorem 4.3. *(Iyama-Yoshino* [10]*) If T_r is an idecomposable direct summand of a cluster-tilting object T, there exists a unique indecomposable direct summand T_r^* of T, not isomorphic to T_r, such that $(T/T_r^*) \oplus T_r^*$ is cluster-tilting.*

In this case we denote $(T/T_r^*) \oplus T_r^*$ by $\mu_r(T)$ and we say that it is the mutation of T at r.

Theorem 4.4. *(Buan-Iyama-Reiten-Scott* [4]*)*

$$Q_{\mu_r T} = \mu_r(Q_T).$$

Definition 4.5. A *cluster-tilting sequence* is a sequence $\mathcal{T}' = (T'_1\ , \ \dots\ , \ T'_n)$ of pairwise non-isomorphic indecomposable objects of $\mathcal{C}_\Gamma$ whose direct sum is a cluster-tilting object T' whose associated quiver $Q_{T'}$ does not have loops or 2-cycles.

Example 4.6. Associated to the vertices $1\ , \ \dots\ , \ n$ of Q we have a complete set of orthogonal idempotents $e_1\ , \ \dots\ , \ e_n$ of Γ. The image $\mathcal{T} = (T_1, \dots, T_n)$ of $(e_1\Gamma, \dots, e_n\Gamma)$ in $\mathcal{C}_\Gamma$ yields the so-called *canonical* cluster-tilting sequence.

Iyama-Yoshino mutation defines a partially defined mutation operation on the cluster-tilting sequences. We use this in the following

Definition 4.7. A cluster-tilting sequence $\mathcal{T}'$ is *reachable* if it can be obtained from the canonical one by mutating and permutating.

5. Main result

Theorem 5.1. *(Keller-Nicolás* [11]*) There is a canonical bijection*

$$\{\textit{reachable cluster collection}\}/\,\mathsf{Braid}(Q) \xrightarrow{\sim} \{\textit{reachable cluster-tilting sequences}\}$$

compatible with mutations and permutations, and preserving the quivers.

6. Main ingredient of the proof

Let Γ be a dg k-algebra such that:

a) it is *homologically non-positive*, *i.e.* $H^p\Gamma = 0$ for $p \geq 1$,
b) $H^p\Gamma$ has finite dimension for each $p \in \mathbf{Z}$,
c) it is *homologically smooth*, *i.e.* Γ is compact in $\mathcal{D}(\Gamma \otimes_k \Gamma^{op})$.

Example 6.1. Γ can be:

a) a complete Ginzburg algebra as in § 2,
b) a finite-dimensional algebra with finite global dimension.

Remark 6.2. Note that conditions a), b) and c) above are preserved under derived Morita equivalence.

Theorem 6.3. *There are canonical bijections between the following sets:*

1) *Non-degenerate t-structures t in $\mathcal{D}\Gamma$ such that the corresponding homological functors $H^{t=n} : \mathcal{D}\Gamma \to \mathcal{D}\Gamma$, $n \in \mathbf{Z}$, preserve products and coproducts, and such that the heart $\mathcal{H}(t)$ has a finite set of projective generators which are compact in $\mathcal{D}\Gamma$.*
2) *Equivalence classes of subsets $\mathcal{P} = \{P'_1 , \ldots , P'_n\}$ of $\mathsf{per}\,\Gamma$ such that*
   a) $\mathsf{Hom}(P'_i, \Sigma^p P'_j) = 0$ *for $p > 0$;*
   b) $\mathsf{per}\,\Gamma = \mathsf{thick}(P'_1, \ldots , P'_n)$.

   Two such sets $\mathcal{P}$ and $\mathcal{P}'$ are equivalent *if* $\mathsf{add}(\mathcal{P}) = \mathsf{add}(\mathcal{P}')$.
3) *Bounded weight structures on $\mathsf{per}\,\Gamma$ whose heart is the additive closure of a finite set.*
4) *Bounded t-structures on $\mathcal{D}_{fd}\Gamma$ whose heart is a length category with a finite number of simples.*
5) *Families $\mathcal{S}' = \{S'_1, \ldots , S'_n\} \subseteq \mathcal{D}_{fd}\Gamma$ of* simple-minded objects[1]*,* i.e. *such that*
   a) $\mathsf{Hom}(S'_i, S'_j) \cong \begin{cases} 0 \text{ if } i \neq j, \\ k \text{ if } i = j; \end{cases}$
   b) $\mathsf{Hom}(S'_i, \Sigma^p S'_j) = 0$ *for each $p < 0$;*
   c) *$\mathcal{S}'$ generates $\mathcal{D}_{fd}\Gamma$.*

The corresponding weight structures of 2) and t-structures of 3) are orthogonal with respect to Hom*,* i.e.

$$\mathsf{Hom}(X, Y) = 0 \text{ if } X \in (\mathsf{per}\,\Gamma)^{w\leq 0} \text{ and } Y \in (\mathcal{D}_{fd}\Gamma)^{t\geq 1}.$$

Example 6.4. Any cluster collection is a family of simple-minded object.

Definition 6.5. In the situation of § 2, we call *cluster heart* to the heart of a t-structure on $\mathcal{D}_{fd}\Gamma$ corresponding to a cluster collection via the bijection between 4) and 5).

Remark 6.6.
- The bijection between 1) and 2) works in great generality, it is implicit in the work of Bondarko [2], and it has been recently rediscovered by Mendoza, Saenz, Santiago and Souto Salorio [17]. To go from 3) to 2), given a weight structure we take its heart. Conversely, given a set $\mathcal{P}$ as in 1) one takes the following weight structure w:

  $$(\mathsf{per}\,\Gamma)^{w\leq 0} = \{M \in \mathsf{per}\,\Gamma \mid \mathsf{Hom}(\mathcal{P}, \Sigma^p M) = 0 \text{ for each } p \geq 1\}$$

  and $(\mathsf{per}\,\Gamma)^{w\geq 0}$ to be the smallest full subcategory of $\mathsf{per}\,\Gamma$ containing $\mathsf{cosusp}(\mathcal{P})$ and closed under direct summands. The heart is $\mathsf{add}(\mathcal{P})$.
- From 2) to 1,4): a set $\mathcal{P}$ as in 1) induces a t-structure in $\mathcal{D}\Gamma$ which restricts to a t-structure in $\mathcal{D}_{fd}\Gamma$.
- From 4) to 1): we construct ‘by hand’ the injective envelopes of the simple-minded objects, and then we prove that they are in the image of the Nakayama functor $\nu : \mathsf{per}\,\Gamma \to \mathcal{D}_0\Gamma = \mathsf{Tria}_{\mathcal{D}\Gamma}(\mathcal{D}_{fd}\Gamma)$, where νP is defined by

  $$\mathsf{Hom}_{\mathcal{D}_0\Gamma}(M, \nu P) \cong D\,\mathsf{Hom}_{\mathcal{D}\Gamma}(P, M)$$

  for each $M \in \mathcal{D}_0\Gamma$.
- The bijections between 2), 4) and 5) are related to work by König and Yang [13].

[1]This is a terminology due to König and Liu, [12].

- To go from 4) to 5) we just take a set of representatives of the isoclasses of the simple objects of the heart. The bijection between 4) and 5) is related to work of Al-Nofayee [19], and Rickard and Rouquier [22]. In our approach we prove the theorem below on compactly generated weight structures, related to work of Pauksztello [20, 21].

Theorem 6.7. *Suppose that $\mathcal{T}$ is a cocomplete triangulated category and that $\mathcal{S}$ is a full additive subcategory stable under direct summands such that:*

a) *$\mathcal{S}$ compactly generates $\mathcal{T}$;*
b) *we have $\mathcal{T}(L, \Sigma^p M) = 0$ for all L and M in $\mathcal{S}$ and all integers $p < 0$;*
c) *the category $\mathsf{Mod}\,\mathcal{S}$ of additive functors $\mathcal{S}^{op} \to \mathsf{Mod}\,\mathbf{Z}$ is semi-simple.*

For X in $\mathcal{T}$ and $p \in \mathbf{Z}$, we write $H^p(X)$ for the object $L \mapsto \mathcal{T}(L, \Sigma^p X)$ of $\mathsf{Mod}\,\mathcal{S}$. Then we have:

1) *There is a unique weight structure $(\mathcal{T}^{>0}, \mathcal{T}^{\leq 0})$ on $\mathcal{T}$ such that $\mathcal{T}^{\leq 0}$ is formed by the objects X with $H^p(X) = 0$ for all $p > 0$ and $\mathcal{T}^{>0}$ is formed by the objects X with $H^p(X) = 0$ for all $p \leq 0$.*
2) *For each object X, there is a truncation triangle*

$$\sigma_{>0}(X) \to X \to \sigma_{\leq 0}X \to \Sigma\sigma_{>0}(X)$$

such that the morphism $X \to \sigma_{\leq 0}X$ induces an isomorphism in H^p for $p \leq 0$ and the morphism $\sigma_{>0}(X) \to X$ induces an isomorphism in H^p for $p > 0$.

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Pedro Nicolás Zaragoza, Facultad de Educación, Universidad de Murcia, Campus de Espinardo, CP: 30100, Murcia, ESPAÑA
E-mail address: pedronz@um.es