Equal images modules and Auslander-Reiten theory for generalized Beilinson algebras

Julia Worch

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We confine our investigations to elementary abelian p-groups.

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$\rightsquigarrow$ kE_r is generated by the elements $x_i := X_i + (X_1^p, \ldots, X_r^p)$.

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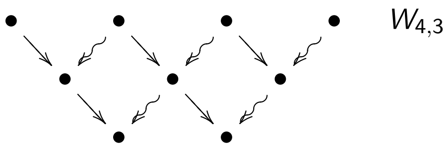

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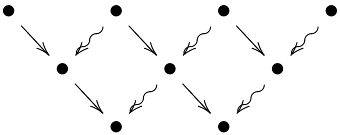

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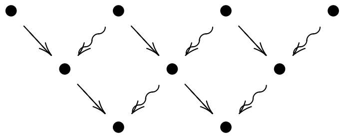

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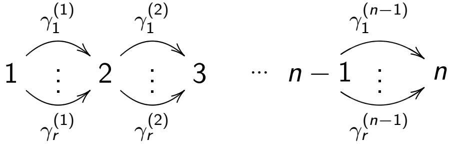

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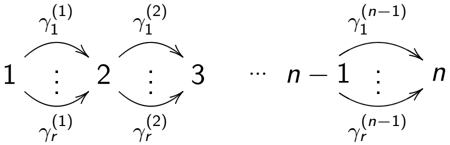

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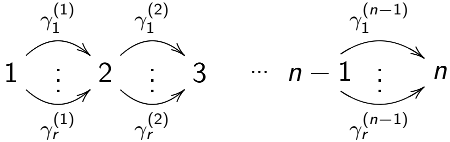

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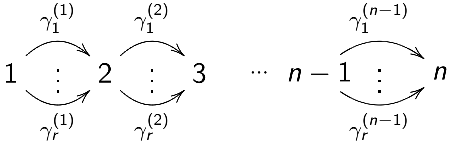

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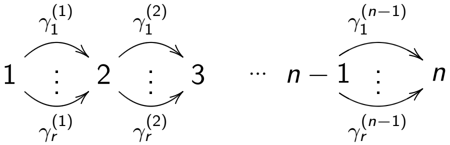

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Let $\Gamma(n, r)$ denote the Auslander-Reiten quiver of $B(n, r)$.

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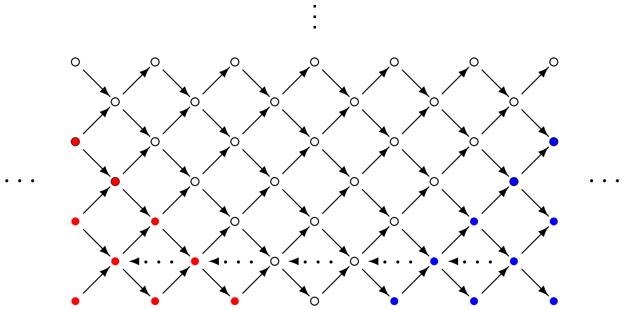

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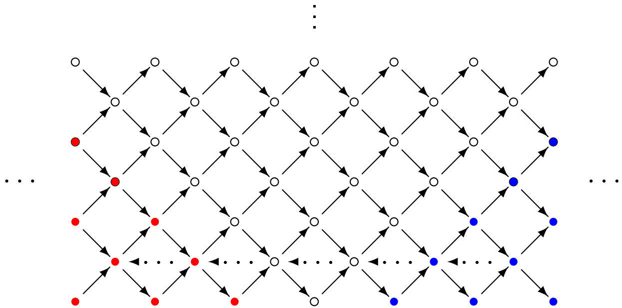

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$\mathbb{Z}A_\infty$-components of $\Gamma(n,r)$

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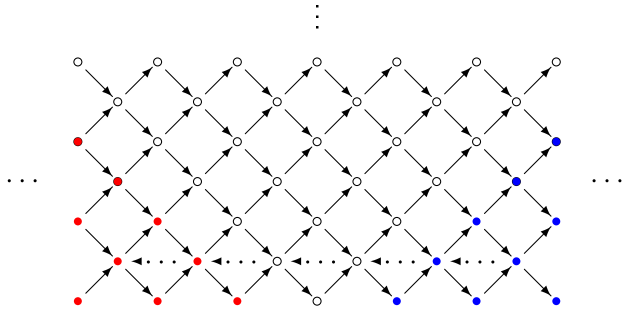

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- $\mathcal{W}(\mathcal{C})=0\Rightarrow\mathcal{C}\subseteq\mathrm{CJT}(n,r)$.

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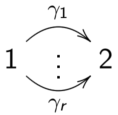

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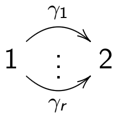

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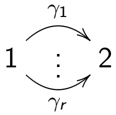

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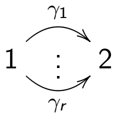

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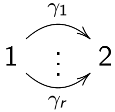

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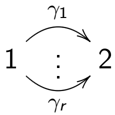

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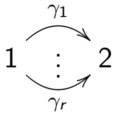

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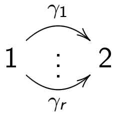

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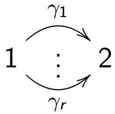

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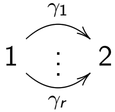

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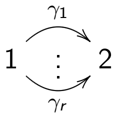

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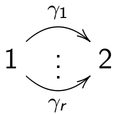

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Let $r > 2, m \geq 3$. The $\mathbb{Z}A_\infty$-component $\mathcal{C}_m$ containing $W_{m,2}^{(r)}$, satisfies $\mathcal{W}(\mathcal{C}_m) = 0$. In particular, $\mathcal{C}_m \subseteq \mathrm{CJT}(2, r)$.

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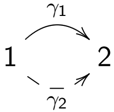
- $r = 2$: extending 

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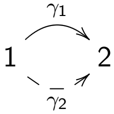
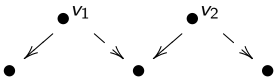
- $r = 2$: extending 

by 

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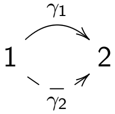
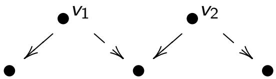
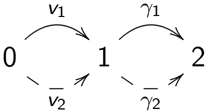
- $r = 2$: extending 

by 

  yields 

with relations $\gamma_2.v_1 = \gamma_1.v_2$.

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Lifting of almost split sequences

Let A be an algebra, $M \in \operatorname{mod} A \rightsquigarrow A[M] = \begin{pmatrix} A & M \\ 0 & k \end{pmatrix}$.

Lemma (Ringel)

Let A be an algebra, M an A-module. Let furthermore

$$0 \to \tau(N) \to E \to N \to 0$$

be an Auslander-Reiten sequence in $\operatorname{mod} A$. *Then*

$$0 \to \begin{pmatrix} \tau(N) \\ \operatorname{Hom}_A(M, \tau(N)) \end{pmatrix} \to \begin{pmatrix} E \\ \operatorname{Hom}_A(M, \tau(N)) \end{pmatrix} \to \begin{pmatrix} N \\ 0 \end{pmatrix} \to 0$$

is an Auslander-Reiten sequence in $\operatorname{mod} A[M]$, *where* $m.\varphi = \varphi(m)$ *for* $m \in M$, $\varphi \in \operatorname{Hom}_A(M, \tau(N))$.