Soergel bimodules and Kazhdan-Lusztig conjectures II

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Joint work with Ben Elias (MIT).
DFG Schwerpunkttagung, Bad Boll, March 27., 2013.

Recall that the Hecke algebra H has a special basis $\{\underline{H}_x \mid x \in W\}$, the *Kazhdan-Lusztig basis*.

Our goal is to understand the Kazhdan-Lusztig positivity conjectures:

$$\underline{H}_x = \sum h_{y,x} H_y \quad h_{y,x} \in \mathbb{Z}_{\geqslant 0}[v]$$
$$\underline{H}_x \underline{H}_y = \sum \mu_{x,y}^z \underline{H}_z \quad \mu_{x,y}^z \in \mathbb{Z}_{\geqslant 0}[v^{\pm 1}].$$

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(There are other positivity conjectures which we ignore here.)

Recall that Soergel introduced a monoidal category $\mathcal{B}$ of graded bimodules over a polynomial ring and proved:

Soergel's categorification theorem (2005):

The split Grothendieck group of $\mathcal{B}$ is isomorphic to the Hecke algebra:

$$\mathrm{ch} : K_0^{\mathrm{split}}(\mathcal{B}) \xrightarrow{\sim} H$$

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I mentioned last time that the key to this conjecture is to prove that B_x have interesting $\mathbb{R}$-Hodge structures. Today I will try to make this precise.

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Hence we normalize so that 0 is the "mirror" or Poincaré duality:

$$H^{-i} \cong (H^i)^* \quad \text{for } i \in \mathbb{Z}.$$

Given any $2n$ dimensional manifold M we could obtain such an H by setting $H^i := H^{i+n}(M; \mathbb{R})$ and

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(Actually, not quite: the intersection pairing on a manifold is graded symmetric. In order to get an H as above we should assume that the odd cohomology of M vanishes.)

Now we introduce Lefschetz operators, which means that most manifolds will fail to provide examples.

A *Lefschetz operator* is a map $L : H \to H$ of degree two, such that

$$\langle L\alpha, \beta \rangle = \langle \alpha, L\beta \rangle$$

for all $\alpha, \beta \in H$.

Example: If M is as above, then multiplication by any two degree class $H^2(M, \mathbb{R})$ provides a Lefschetz operator.

A Lefschetz operator L *satisfies the hard Lefschetz theorem* if for all $i \geqslant 0$, i iterates of L provide an isomorphism

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This is equivalent to L being the e in an $\mathfrak{sl}_2(\mathbb{R})$-action on H with h equal to the "degree operator": $h\alpha = m\alpha$ for all $\alpha \in H^m$.

The main example of operators satisfying the hard Lefschetz theorem is as follows:

Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a projective algebraic variety. Let $\beta \in H^2(X, \mathbb{R})$ be the first Chern class of an ample line bundle on X. Then multiplication by β satisfies the hard Lefschetz theorem.

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This is the “hard Lefschetz theorem”; so-called because it is hard!

A folk history: Lefschetz "proved" the result in his famous book in which every statement is true, and every proof is false. Hodge gave a completely different and complete proof using his theory of harmonic forms. Chern recaste Hodge's proof in terms of the representation theory of $\mathfrak{sl}_2(\mathbb{C})$. Grothendieck christened the theorem the "Theorème de Lefschetz vache": it caused him much suffering. Deligne used his theory of weights to make Lefschetz's proof, using Lefschetz pencils, rigourous.

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Lefschetz's "other" theorem is the weak Lefschetz theorem: "almost all cohomology is contained in a hyperplane section". This is a much easier statement.

Now we return to our axiomatic setting. Assume that L satisfies the hard Lefschetz theorem.

We have two different structures: L^i identifies H^{-i} and H^i, and $\langle -, - \rangle$ pairs them. Putting these two structures together we get the *Lefschetz forms*:

$$(\alpha, \beta)_L^{-i} := \langle \alpha, L^i \beta \rangle \quad \text{for all } \alpha, \beta \in H^{-i}.$$

Because L satisfies the hard Lefschetz theorem we have a primitive decomposition

$$H = \bigoplus_{i \geqslant 0} \left(\bigoplus_{i \geqslant j \geqslant 0} L^j P_L^{-i} \right)$$

where

$$P_L^{-i} := \ker L^{i+1} \subset H^{-i}$$

is the *primitive subspace* or "lowest weight vectors".

Now assume that $H^{\mathrm{odd}} = 0$ or $H^{\mathrm{even}} = 0$. (H is *parity*).

In a fixed negative degree the primitive decomposition looks as follows:

$$H^{-i} = P_L^{-i} \oplus L P_L^{-i-2} \oplus \cdots \oplus L^{-\min/2} P_L^{\min}.$$

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Exercise: This decomposition is orthogonal with respect to the Lefschetz form $(-,-)_L^{-i}$.

We say that L *satisfies the Hodge-Riemann bilinear relations* if the restriction of $(-,-)_L^{-i}$ to $L^i P_L^{\min+2i}$ is $(-1)^i$-definite.

In other words, the signs of $(-,-)_L^{-i}$ alternate in the above decomposition.

A side note:

The definiteness in the Hodge-Riemann bilinear relations plays a key role in our proof. This rich source of information disappears completely if we pass to $\mathbb{C}$.

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Also an understanding of these forms over $\mathbb{Z}$ (in particular their non-degeneracy) is closely linked to Lusztig's modular conjecture.

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The definiteness in the Hodge-Riemann bilinear relations plays a key role in our proof. This rich source of information disappears completely if we pass to $\mathbb{C}$.

Also an understanding of these forms over $\mathbb{Z}$ (in particular their non-degeneracy) is closely linked to Lusztig's modular conjecture.

We suspect (but haven't checked) that Soergel's conjecture fails for certain representations of affine Weyl groups which admit no $\mathbb{R}$-forms! Hence the appearance of $\mathbb{R}$ might be an essential feature, and not a means to an end. Perhaps this also explains why complex reflection groups don't have a good Soergel bimodule theory.

We now state the main results:

We let $\mathfrak{h}$ be a real reflection representation of W and an element $\rho \in \mathfrak{h}^*$ having the property that

$$\langle w(\rho), \alpha_s^\vee \rangle > 0 \Leftrightarrow sw > w.$$

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For example, it is always possible to find such a ρ if $\mathfrak{h}$ is the geometric representation.

One should think that ρ is the "class of an ample line bundle", whether or not this makes sense.

Example: If $\mathfrak{h}$ is the Cartan subalgebra of a complex semi-simple Lie algebra $\mathfrak{g}$ and Δ is a fixed choice of simple roots (determining the simple reflections $S \subset W$) in the Weyl group of W then $\gamma \in \mathfrak{h}^*$ satisfies the above condition if and only if γ is "strictly dominant":

$$\langle \gamma, \alpha^\vee \rangle > 0 \quad \text{for all } \alpha \in \Delta$$

If γ is in addition integral then this is the condition for γ to be the class of an ample line bundle on the corresponding flag variety.

Recall that to our choice of $\mathfrak{h}$ we can associate the category $\mathcal{B}$ of Soergel bimodules:

We consider R the regular functions on $\mathfrak{h}$ (also known as the symmetric algebra on $\mathfrak{h}^*$). This is simply a multivariate polynomial ring over $\mathbb{R}$. We grade R so that $\deg \mathfrak{h}^* = 2$. Because W acts on $\mathfrak{h}$ it also acts on R via graded algebra automorphisms.

We consider $R-\mathrm{Bim}$ the monoidal category of R-bimodules and abbreviate $MM' = M \otimes_R M'$.

For $s \in S$ let $B_s := R \otimes_{R^s} R(1)$. Consider

$$\mathcal{B} = \begin{array}{c} \text{full Karoubian subcategory of } R-\text{Bim} \\ \text{generated by } B_s(m) \text{ for all } s \in S,\ m \in \mathbb{Z}. \end{array}$$

In other words, the objects $\mathcal{B}$ are the graded R-bimodule direct summands of bimodules of the form

$$B_s B_t \dots B_u = R \otimes_{R^s} R \otimes_{R^u} R \otimes \dots \otimes_{R^u} R(m)$$

for arbitrary sequences $st \dots u$ and $m \in \mathbb{Z}$.

As we discussed last time, Soergel showed that the indecomposable Soergel bimodules are classified by $x \in W$. Given any expression $\underline{w} = st \dots u$ of length m let

$$B_{\underline{w}} := B_s B_t \dots B_u = R \otimes_{R^s} R \otimes_{R^t} \dot{\otimes}_{R^u} R(m).$$

This is a *Bott-Samelson bimodule*.

Then B_x may be characterised as the unique direct summand of

$$B_{\underline{x}} := B_s B_t \dots B_u$$

where $\underline{x} = st \dots u$ is a reduced expression for x, which does not occur as a direct summand of a shift of $B_{\underline{w}}$ for any expression $\underline{w}$ of length strictly less than m.

Example: Remember the situation for $W = S_3$ and $R = \mathbb{R}[X_1, X_2, X_3]$. We saw that

$$B_{id} = R, B_s = B_s, B_t = B_t, B_{st} = B_s B_t, B_{ts} = B_t B_s$$

and

$$B_s B_t B_s = B_{sts} \oplus B_s \quad B_t B_s B_t = B_{sts} \oplus B_t.$$

and we have: $B_{sts} = (R \otimes_{R^{s,t}} R)(3)$.

Recall that B_x is an R-bimodule. A key to the proof is to understand the "Soergel modules":

$$\overline{B_x} := B_x \otimes_R \mathbb{R}.$$

whereas B_x is an infinite dimensional R-bimodule, $\overline{B_x}$ is a finite dimensional self-dual graded $\mathbb{R}$-vector space.

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In situations where one "has geometry" the B_x correspond to equivariant (intersection) cohomology whereas $\overline{B_x}$ corresponds to ordinary (intersection) cohomology.

Question: What are the Hodge-Riemann bilinear relations in equivariant cohomology? Is there a useful theory waiting here?

As always, one should keep in mind the case when W is finite with longest element w_0. Then

$$B_{w_0} = R \otimes_{R^W} R(\ell(w_0))$$

and

$$B_{w_0} = R/(R^W_+)(\ell(w_0))$$

is the coinvariant ring.

Hence we can realize $B_x \overset{\oplus}{\subset} B_{\underline{x}}$ for a reduced expression $\underline{x}$ for x.

Now $B_{\underline{x}}$ is a graded ring, and is equipped with a symmetric "intersection form"

$$\langle -, - \rangle : B_{\underline{x}} \times B_{\underline{x}} \to R$$

given by

$$\langle f, g \rangle := \mathrm{Tr}(fg)$$

for some map $\mathrm{Tr} : B_{\underline{x}} \to R$ (take the coefficient of a natural "top" class in a basis for $B_{\underline{x}}$ as a right R-bimodule).

It is easy to see combinatorially that $\langle -, - \rangle$ induces a non-degenerate form

$$\langle -, - \rangle_{\mathbb{R}} : \overline{B_{\underline{x}}} \times \overline{B_{\underline{x}}} \to \mathbb{R}$$

where $\overline{B_{\underline{x}}} = B_{\underline{x}} \otimes_R \mathbb{R}$.

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Hence $\overline{B_x}$ is a finite dimensional $\mathbb{R}$-vector space equipped with non-degenerate symmetric graded form

$$\langle -, - \rangle_{\mathbb{R}} : \overline{B_x} \times \overline{B_x} \to \mathbb{R}.$$

It is obvious by construction that left multiplication by ρ on $\overline{B_x}$ provides a Lefschetz operator (i.e. $\langle \rho b, b' \rangle = \langle b, \rho b' \rangle$ for all $b, b' \in \overline{B_x}$).

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Main theorem:

Left multiplication by ρ on $\overline{B_x}$ induces a Lefschetz operator satisfying the hard Lefschetz theorem and Hodge-Riemann bilinear relations.

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We adapt beautiful ideas of de Cataldo and Migliorini:

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However the adaption is by no means immediate. The biggest problem is finding a substitute for the weak Lefschetz theorem. We notice that the first differential on a Rouquier complex:

$$\overline{B_x} \to \bigoplus_{y<x} \overline{B_y}^{\oplus m_y}$$

provides a substitute for the weak Lefschetz theorem.

This should have other applications!

Here is one key idea, stolen from de Cataldo and Migliorini:

Limit lemma: Suppose that L_ζ is a continuous family of Lefschetz operators parametrized by an interval $I \subset \mathbb{R}$. If all members satisfy the hard Lefschetz theorem, and one satisfies the Hodge-Riemann bilinear relations, then all satisfy the Hodge-Riemann bilinear relations.

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Proof: The Lefschetz forms $(-,-)^{-i}_{L_\zeta}$ are all non-degenerate (by hard Lefschetz) symmetric, and have the right signature for one value of ζ. Hence they always have the right signature, because signatures of symmetric real matrices can't change in families. *qed*

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By considering families of deformed Lefschetz operators one can sometimes make all problems disappear as $\zeta \to \infty$.

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Because a definite form restricted to a subspace stays definite these "local intersection forms" are definite, and hence non-degenerate.

This reduces the problem to establishing the Hodge-Riemann bilinear relations for $\overline{B_x B_s}$.

Other applications:

If σ is an automorphism of the Dynkin diagram then W^{σ} is a Coxeter group, with simple generators given by the products of orbits of σ on the simple reflections.

One obtains in this way a "Hecke algebra with unequal parameters". It has a Kazhdan-Lusztig basis, however the Kazhdan-Lusztig polynomials are no longer positive.

(The study of these polynomials is important when one considers quasi-split finite reductive algebraic.)

We can prove all of Lusztig's conjectures about Kazhdan-Lusztig basis for unequal parameter Hecke algebras which come from diagram automorphisms. The basic idea is to realize the unequal parameter Hecke algebra as the equivariant K-theory of the category of Soergel bimodules attached to W. The Kazhdan-Lusztig polynomials emerge as the traces of σ on hom spaces. (All of this is a straightforward adaption of ideas of Lusztig.)

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We need Soergel's conjecture to prove that certain idempotents behave nicely.

Recently, Lusztig and Vogan defined a module for the Hecke algebra with basis the (twisted) involutions in W. We can categorify their construction, and prove their conjectures on its basis.

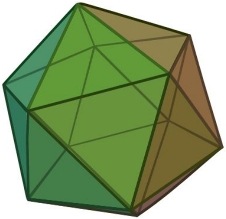
One nice consequence: any Kazhdan-Lusztig polynomial $h_{x,y}$ attached to involutions x and y in any Coxeter group W has a canonical decomposition

$$h_{x,y} = h^+_{x,y} + h^-_{x,y}.$$

where $h^+_{x,y}$ and $h^-_{x,y}$ are positive polynomials.

Twenty lectures and exercises (with Ben Elias) on these ideas:

http://people.mpim-bonn.mpg.de/geordie/aarhus/



Thanks for listening!