Soergel bimodules and Kazhdan-Lusztig conjectures

Geordie Williamson

DFG Schwerpunkttagung, Bad Boll, March 25., 2013.

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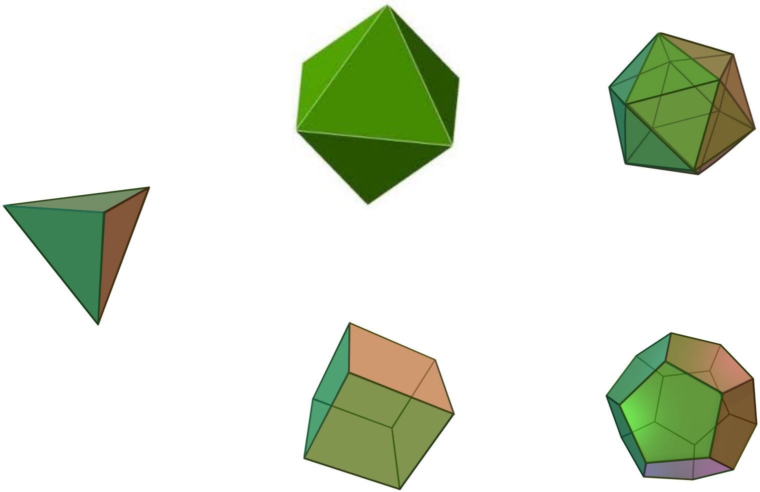
Slogan: Soergel bimodules look like the real cohomology of smooth projective algebraic varieties.

This lecture I will describe what Soergel bimodules are and some ways of thinking about them. Next lecture I will go into more detail and explain precisely what is meant by the slogan.

At the heart of this story are Coxeter groups.

In this talk, W will denote a general Coxeter group. However you will lose nothing by assuming W is the symmetries of one of the platonic solids:



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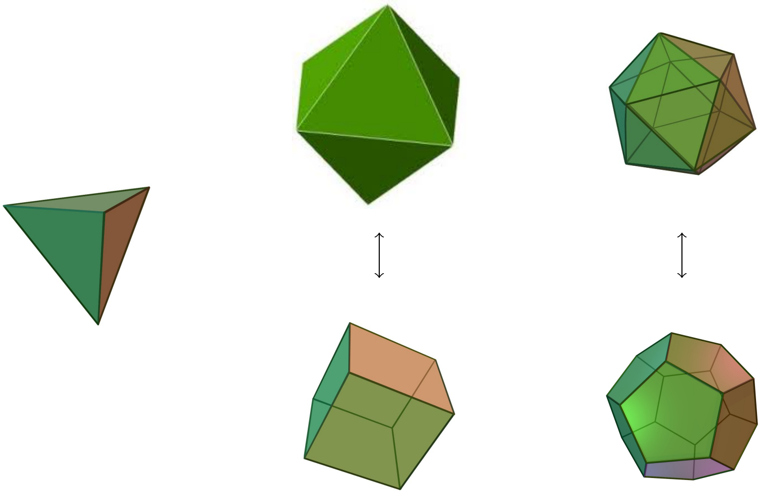

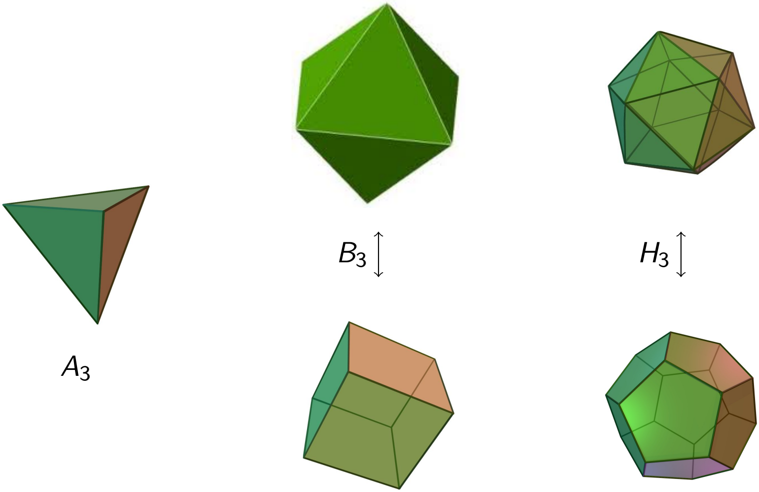
A_3
B_3
H_3

Already when considering the platonic solids an important distinction occurs. The groups A_3 and B_3 are *crystallographic* (that is, stablize a lattice in $\mathbb{R}^3$), whilst the group H_3 is not.

Many examples of finite Coxeter groups are provided by the Weyl groups of compact Lie groups. For example the symmetries of the butterfly is the Weyl group of $SU(2)$, the symmetries of the hexagon or snowflake is the Weyl group of G_2, the symmetries of the tetrahedron is the Weyl group of $SL(4)$ and the symmetries of the octahedron is the Weyl group of $SO(7)$ or $Sp(6)$.

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However it is important for the story that not all Coxeter groups (for example the icosahedral group H_3) belong to Lie groups.

Even simpler examples of this phenomenon is provided by the dihedral groups: only the symmetries of the triangle, square and hexagon occur as Weyl groups.

Throughout (W, S) will denote a Coxeter system:

$$\begin{aligned} W &= \langle s \in S \mid s^2 = 1, (st)^{m_{st}} = 1 \rangle \\ &= \langle s \in S \mid s^2 = 1, \underbrace{st\ldots}_{m_{st}\text{terms}} = \underbrace{ts\ldots}_{m_{st}\text{terms}} \rangle \end{aligned}$$

(where $m_{st} \in \{2, 3, \ldots, \infty\}$).

For example, we could take W to be a real reflection group.

To a Coxeter system (W, S) one may associate a simplicial complex $CC(W)$ called the Coxeter complex of W.

Let $n = |S|$ denote the rank of W. Its construction is as follows:

- colour the n faces of the standard $n - 1$-simplex by the set S,
- take one such simplex for each element $w \in W$,
- glue the simplex corresponding to w to that corresponding to ws along the wall coloured by s.

For example, consider $W = S_3$:

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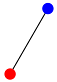

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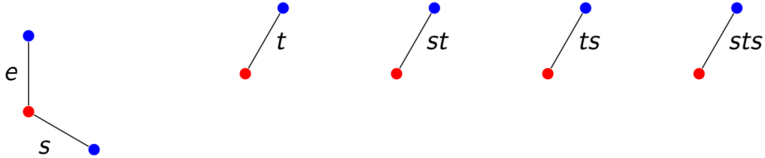

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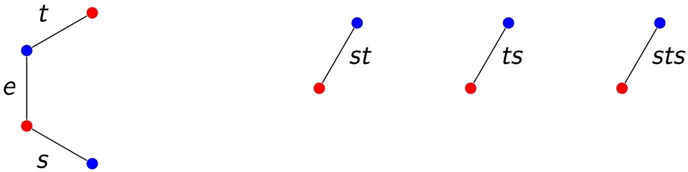

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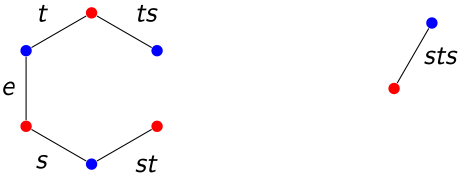

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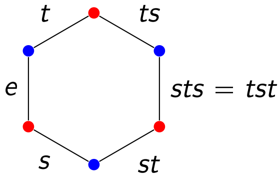

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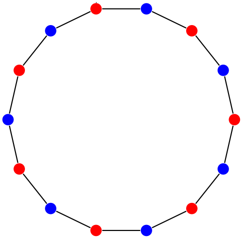

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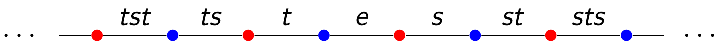

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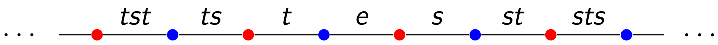

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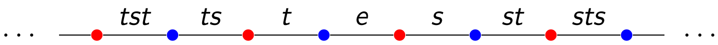
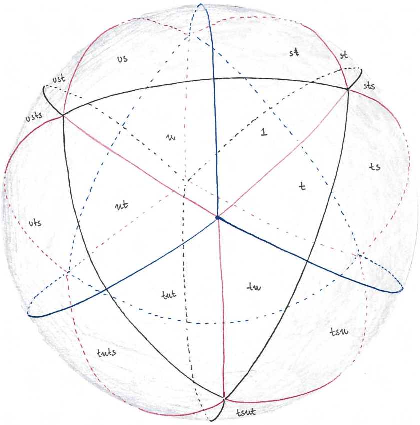


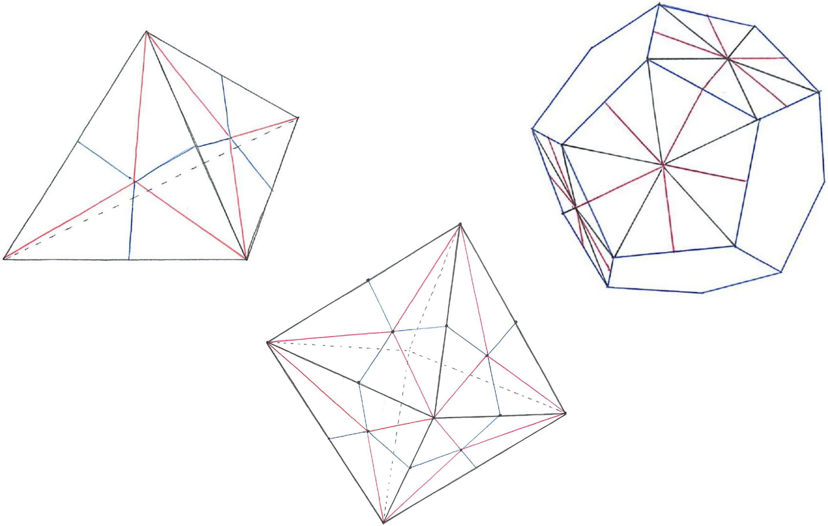
Any Coxeter complex for a rank 2 Coxeter group is homeomorphic to a circle or the real line.

This fact can be used to "explain" the presentation of W.

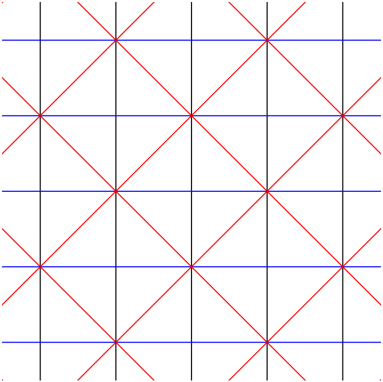
The Coxeter complex of S_4 =  :

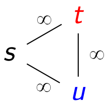


(barycentric subdivision of the tetrahedron).



$$s \overset{4}{\text{———}} t \overset{4}{\text{———}} u$$

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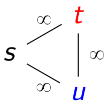
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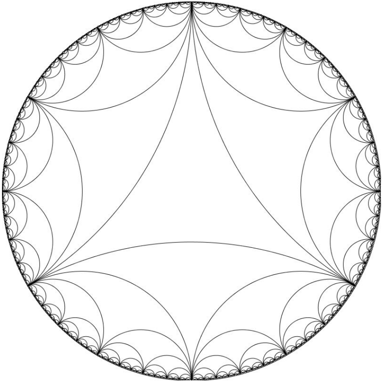

;

By construction $|(W,S)|$ has a left action of W.

W also acts on the alcoves of $|(W,S)|$ on the right by

$$\Delta_w \cdot s = \Delta_{ws}.$$

This action is *not* simplicial, but is “local”: cross the wall coloured by s.

The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W. Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W. The product in W induces a multipliction in $\mathbb{Z}W$.

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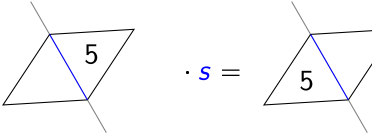
Hence we can picture an element of $\mathbb{Z}W$ as the assignment of integers to each alcove, such that only finitely many are non-zero. If we view $\mathbb{Z}W$ as a right module over itself it is easy to picture the action of the elements of S:



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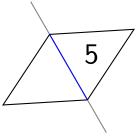
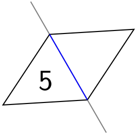
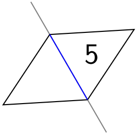


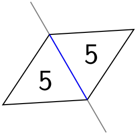
$\cdot s =$



Similarly ("s averaging operator")



$\cdot (1 + s) =$



Let $\ell : W \to \mathbb{N}$ denote the length function on W:

$$\begin{aligned}\ell(w) &= \text{length of a minimal expression for } w \text{ in the generators } s \\ &= \text{number of walls crossed in a minimal path } id \to w \text{ in } |(W,S)|.\end{aligned}$$

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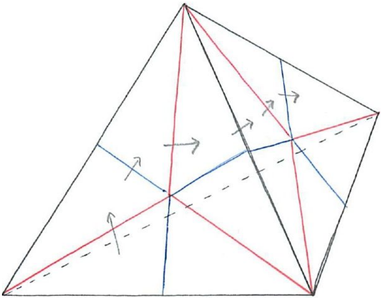

The Hecke algebra H is a quantization of $\mathbb{Z}W$. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with basis $\{H_x \mid x \in W\}$ parametrised by W. If we write $\underline{H}_s := H_s + vH_{id}$ then the multiplication in H is determined by

$$H_x \underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } \ell(xs) > \ell(x), \\ H_{xs} + v^{-1}H_x & \text{if } \ell(xs) < \ell(x). \end{cases}$$

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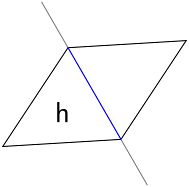
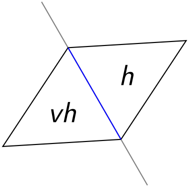
We can visualise this as follows: ("quantized averaging operator")

id



$\cdot \underline{H}_s =$



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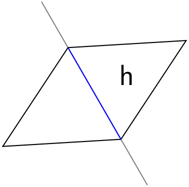

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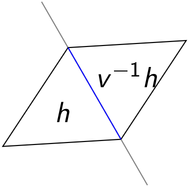
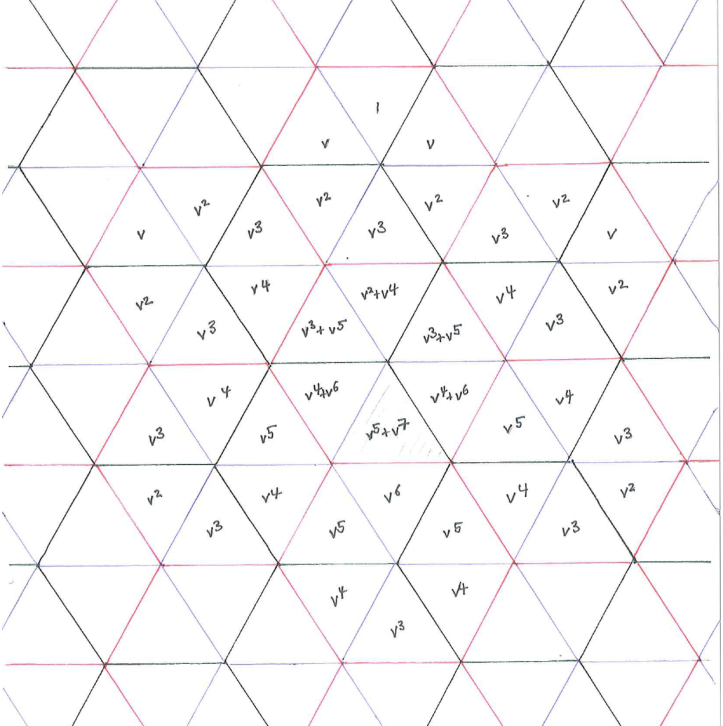


In 1979 Kazhdan and Lusztig defined a new basis for the Hecke algebra using the combinatorial structure of W. We denote this new basis by $\{\underline{H}_x \mid x \in W\}$. It satisfies

$$\underline{H}_x := H_x + \sum_{\substack{y \in W \\ \ell(y) < \ell(x)}} h_{y,x} H_y$$

with $h_{y,x} \in v\mathbb{Z}[v]$. These polynomials are the *Kazhdan-Lusztig polynomials*.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v^4

The definition is inductive. The first few Kazhdan-Lusztig basis elements are easily defined:

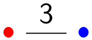
$$\underline{H}_{id} := H_{id}, \quad \underline{H}_s := H_s + vH_{id} \quad \text{for } s \in S.$$

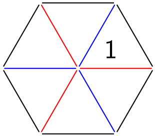
Now the work begins. Suppose that we have calculated $\underline{H}_y$ for all y with $\ell(y) \leqslant \ell(x)$. Choose $s \in S$ with $\ell(xs) > \ell(x)$ and write

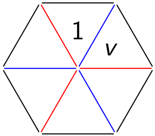
$$\underline{H}_x \underline{H}_s = H_{xs} + \sum_{\ell(y) < \ell(xs)} g_y H_y.$$

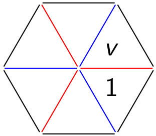
The formula for the action of $\underline{H}_s$ shows that $g_y \in \mathbb{Z}[v]$ for all $y < \ell(xs)$. If all $g_y \in v\mathbb{Z}[v]$ then $\underline{H}_{xs} := \underline{H}_x \underline{H}_s$. Otherwise we set

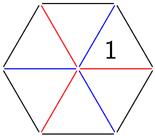
$$\underline{H}_{xs} = \underline{H}_x \underline{H}_s - \sum_{\substack{y \\ \ell(y) < \ell(x)}} g_y(0) \underline{H}_y.$$

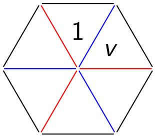


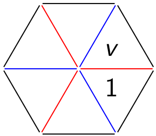
$\underline{H}_{id} =$ 

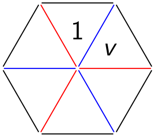
$\underline{H}_{\color{blue}t} =$ 

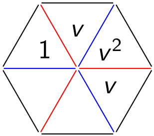
$\underline{H}_{\color{red}s} =$ 

$\underline{H}_{id} =$ 

$\underline{H}_t =$ 

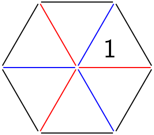
$\underline{H}_s =$ 

$\underline{H}_t \underline{H}_s =$ 

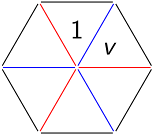
$\cdot \underline{H}_s =$ 

$= \underline{H}_{ts}$

$\underline{H}_{id} =$



$\underline{H}_{t} =$



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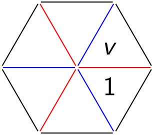

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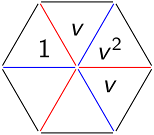
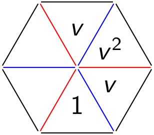
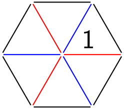
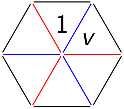
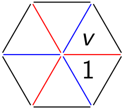


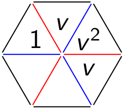
$\underline{H}_{st} =$

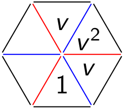


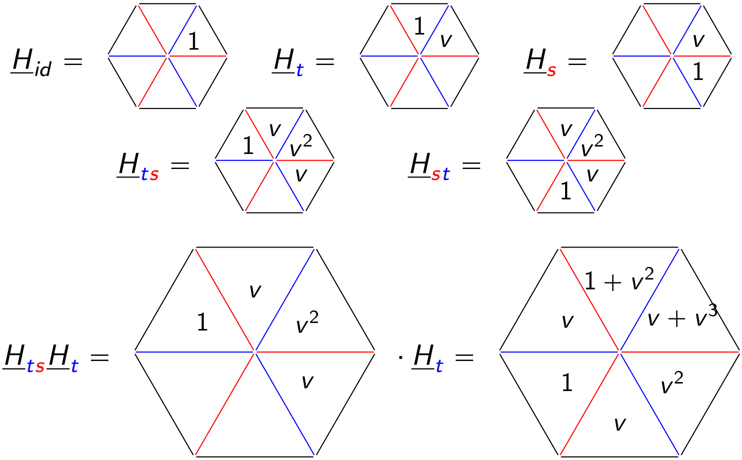
$\underline{H}_{id} =$ 

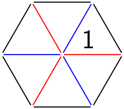
$\underline{H}_{t} =$ 

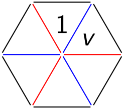
$\underline{H}_{s} =$ 

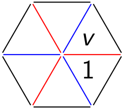
$\underline{H}_{ts} =$ 

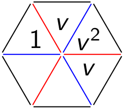
$\underline{H}_{st} =$ 

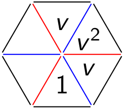

$\underline{H}_{id} =$
1
$\underline{H}_t =$
1
v
$\underline{H}_s =$
v
1
$\underline{H}_{ts} =$
1
v
v^2
v
$\underline{H}_{st} =$
v
v^2
v
1
$\underline{H}_{ts}\underline{H}_t =$
1
v
v^2
v
$\cdot \underline{H}_t =$
$1 + v^2$
v
$v + v^3$
1
v^2
v

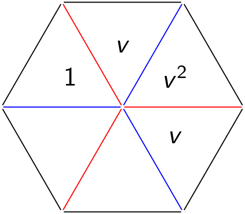
$\underline{H}_{id} =$ 

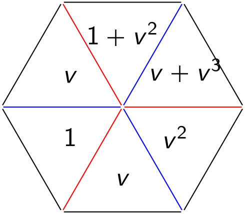
$\underline{H}_t =$ 

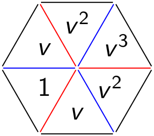
$\underline{H}_s =$ 

$\underline{H}_{ts} =$ 

$\underline{H}_{st} =$ 

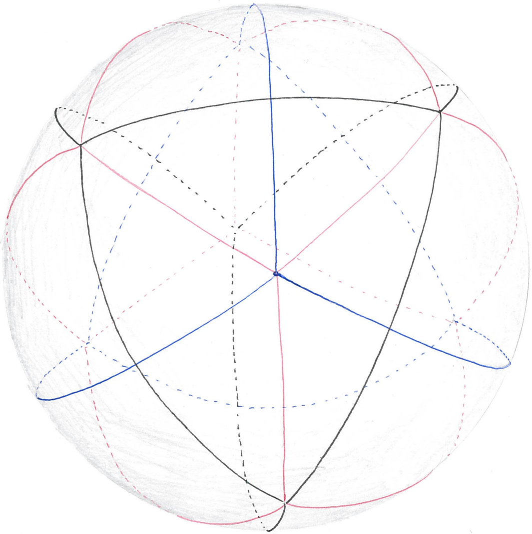
$\underline{H}_{ts}\underline{H}_t =$ 

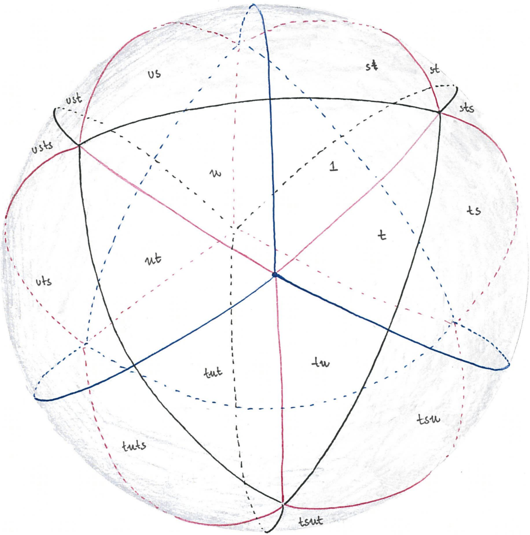
$\cdot\, \underline{H}_t =$ 

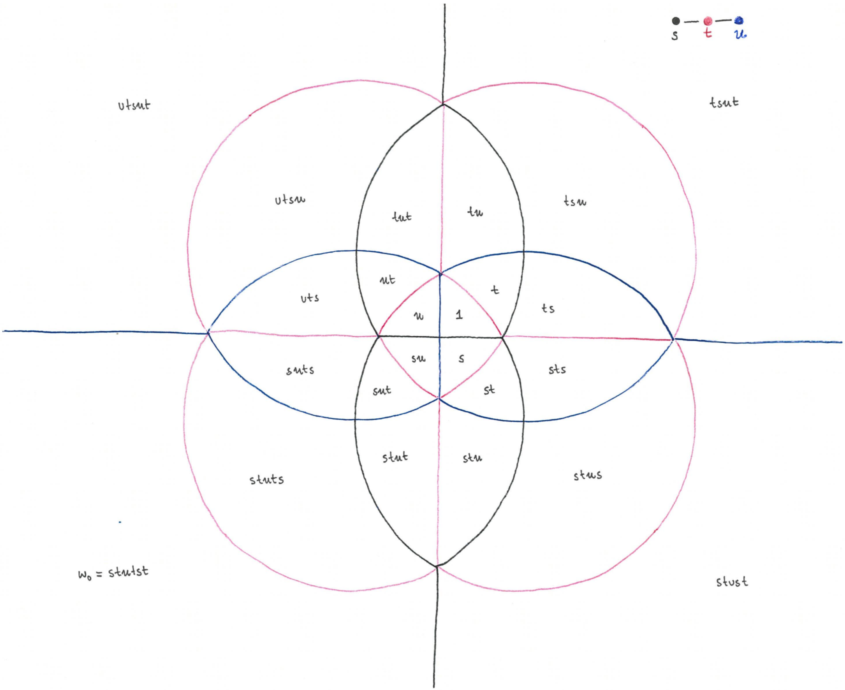
Hence: $\underline{H}_{tst} = \underline{H}_{ts}\underline{H}_t - \underline{H}_t =$ 

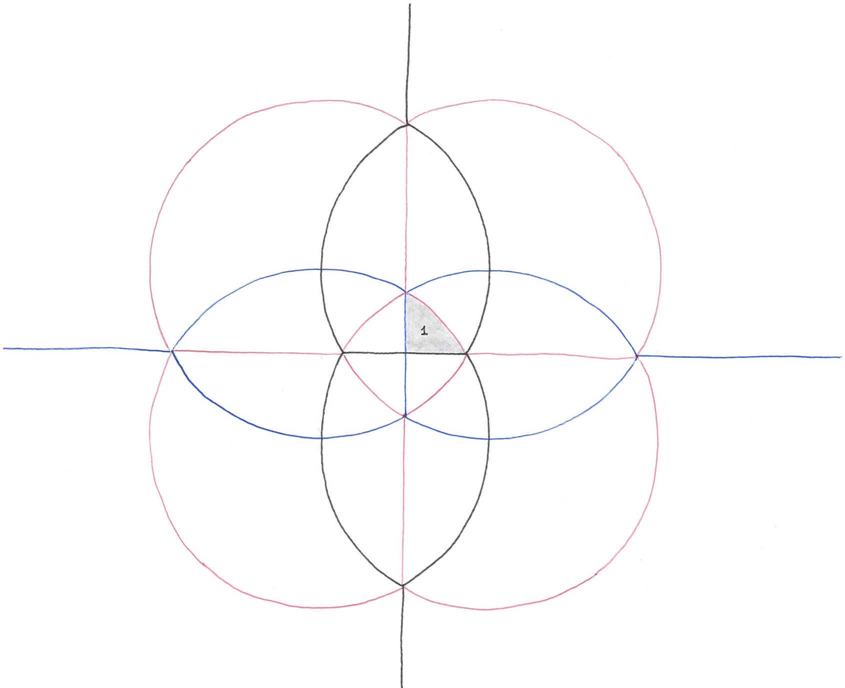
For dihedral groups (rank 2) we always have $h_{y,x} = v^{\ell(x)-\ell(y)}$ (Kazhdan-Lusztig basis elements are *smooth*.)

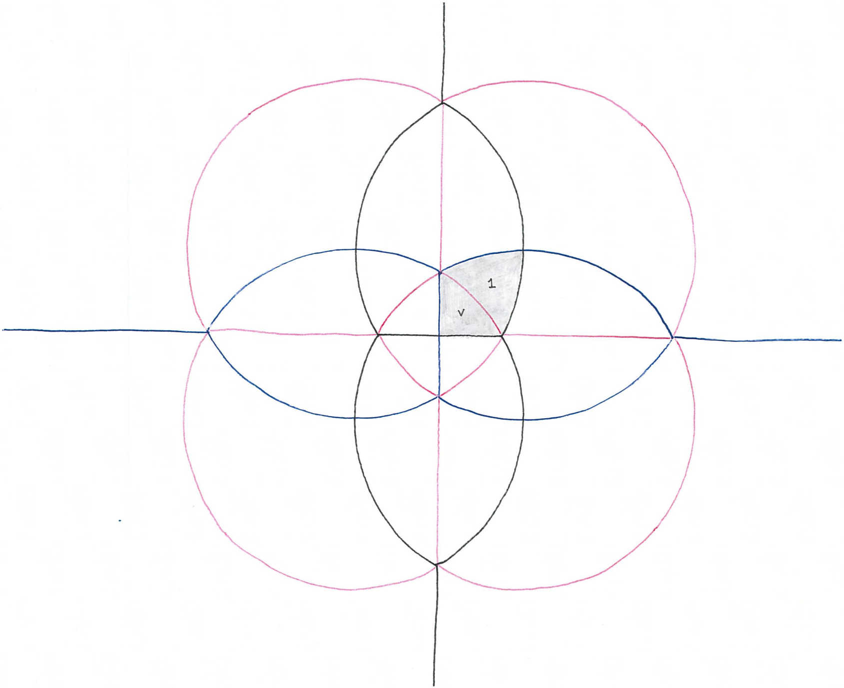
However in higher rank the situation quickly becomes more interesting...

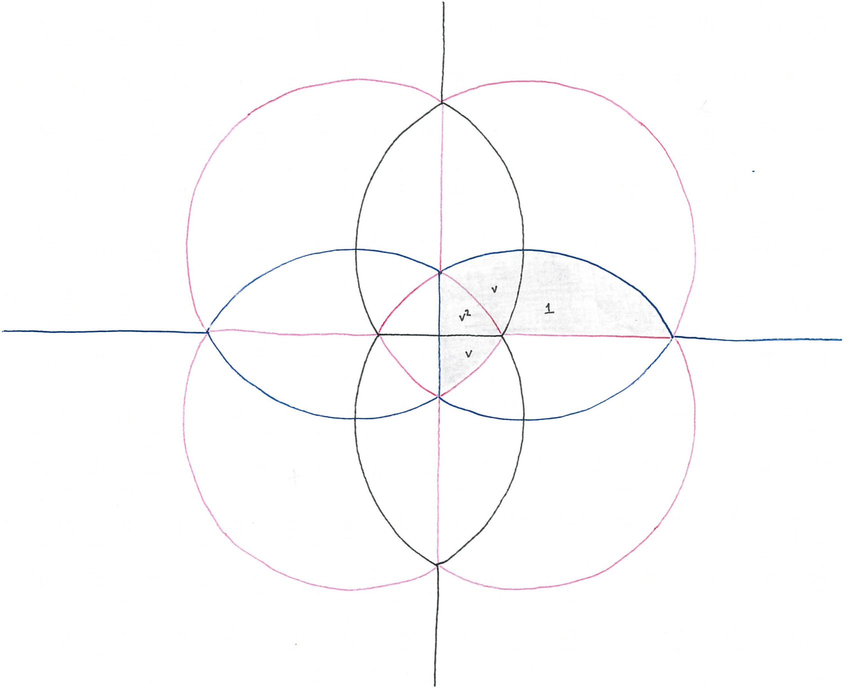


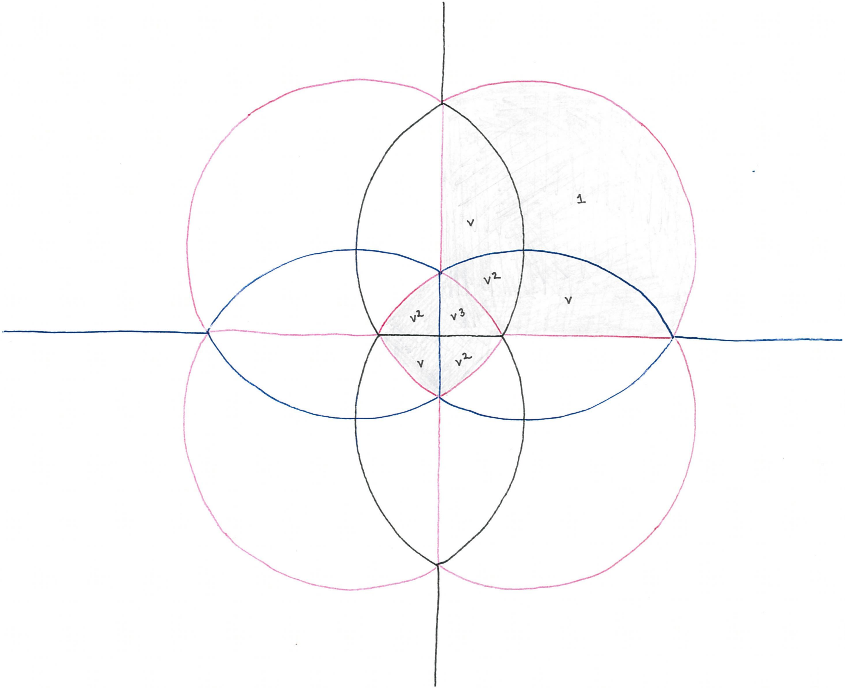

us
st
st
ust
sts
usts
u
1
ts
t
ut
uts
tut
tu
tsu
tuts
tsut


s
t
u
utsut
tsut
utsu
tut
tu
tsu
ut
t
uts
u
1
ts
suts
su
s
sts
sut
st
stut
stu
stus
stuts
w0 = stutst
stust

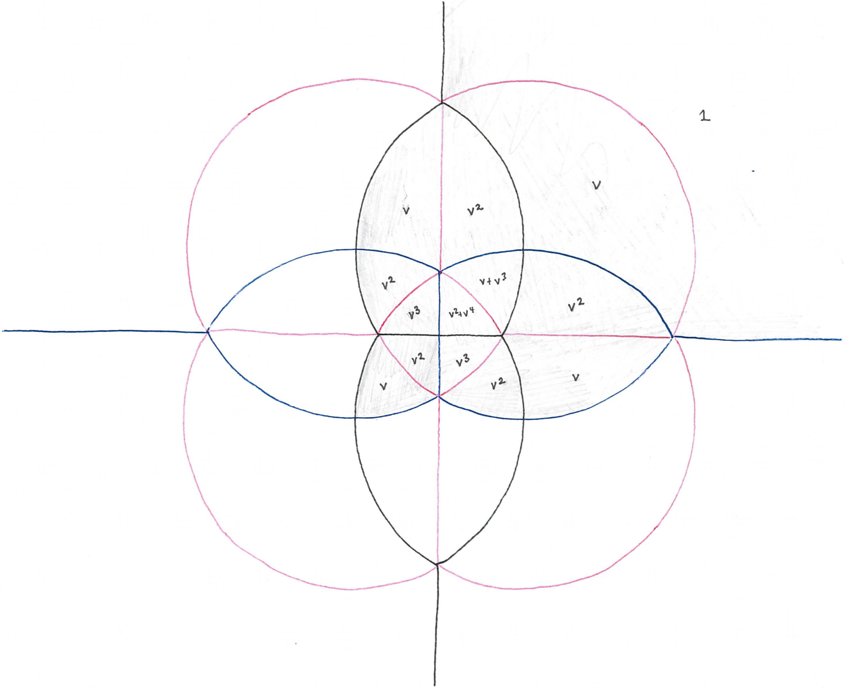

1

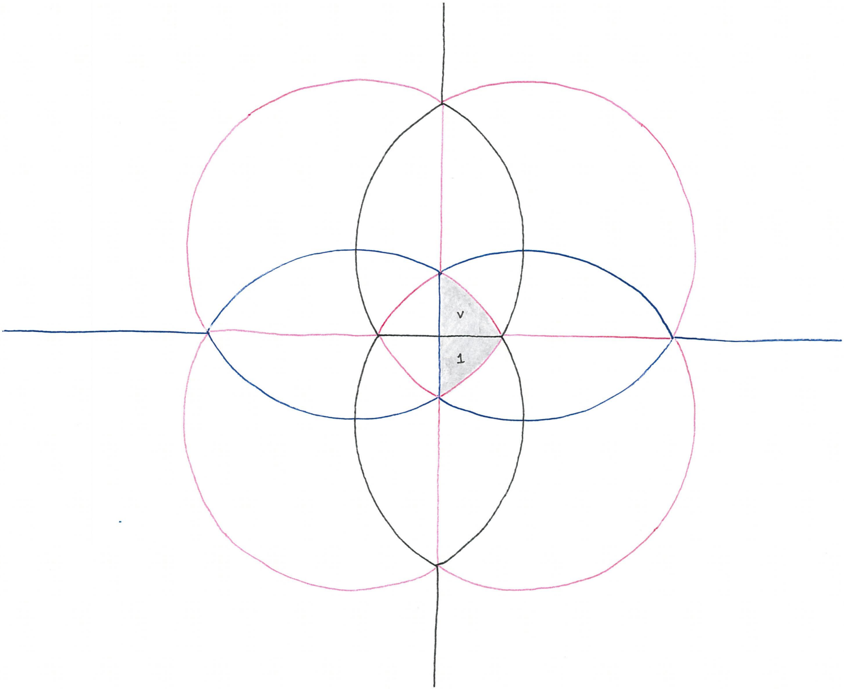

1
v

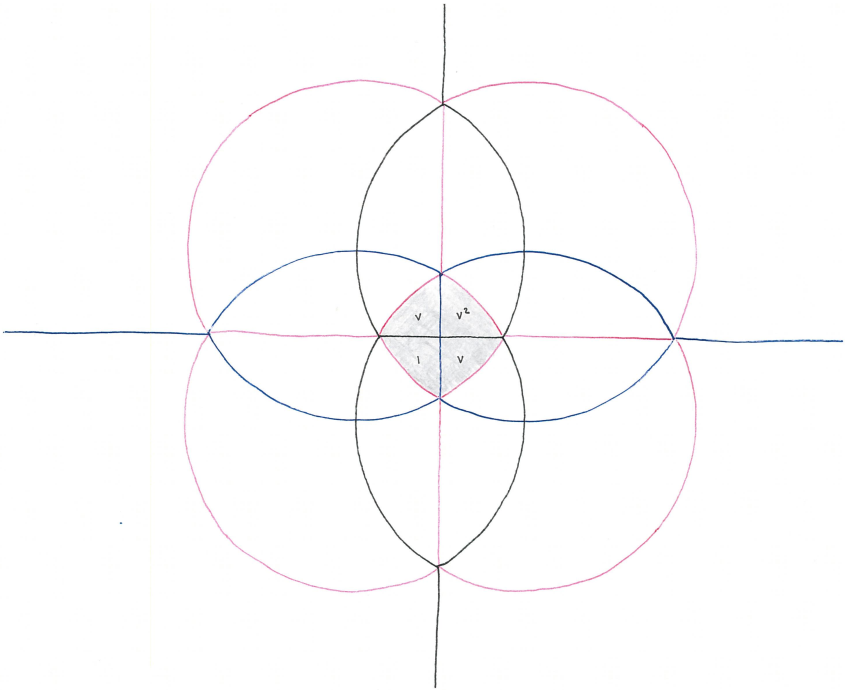

v
v²
1
v

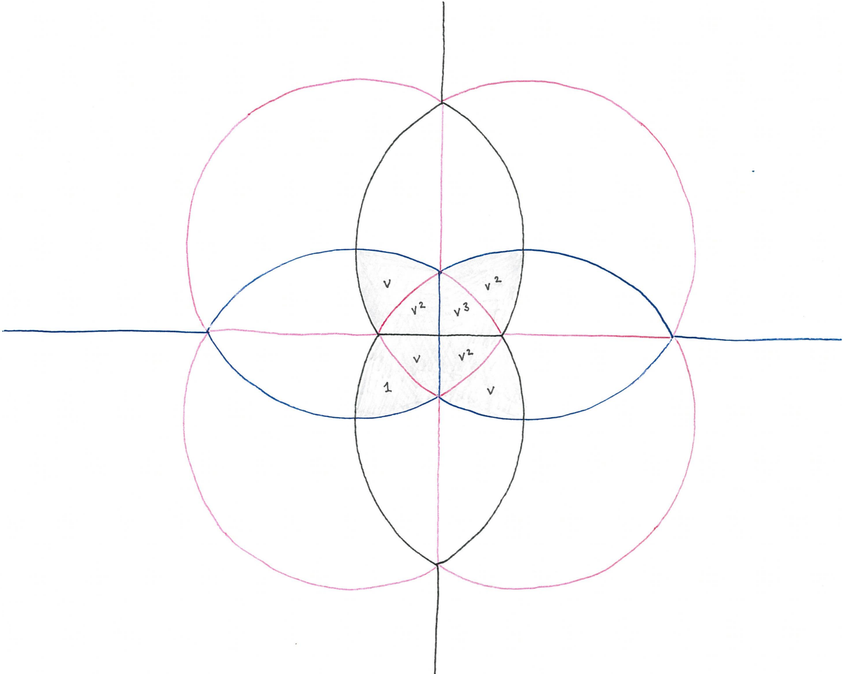

1
v
v2
v
v2
v3
v
v2

1

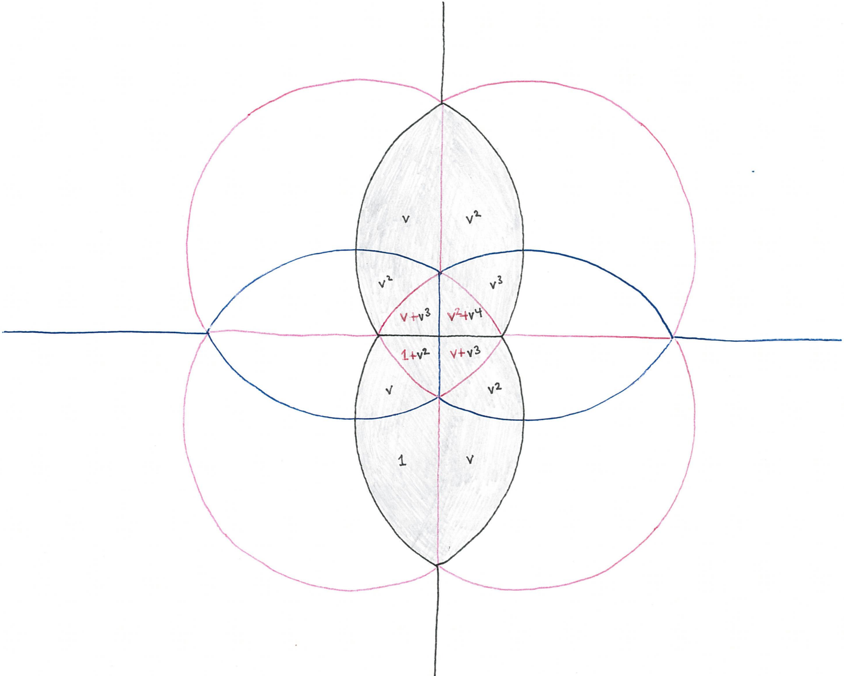



v
1

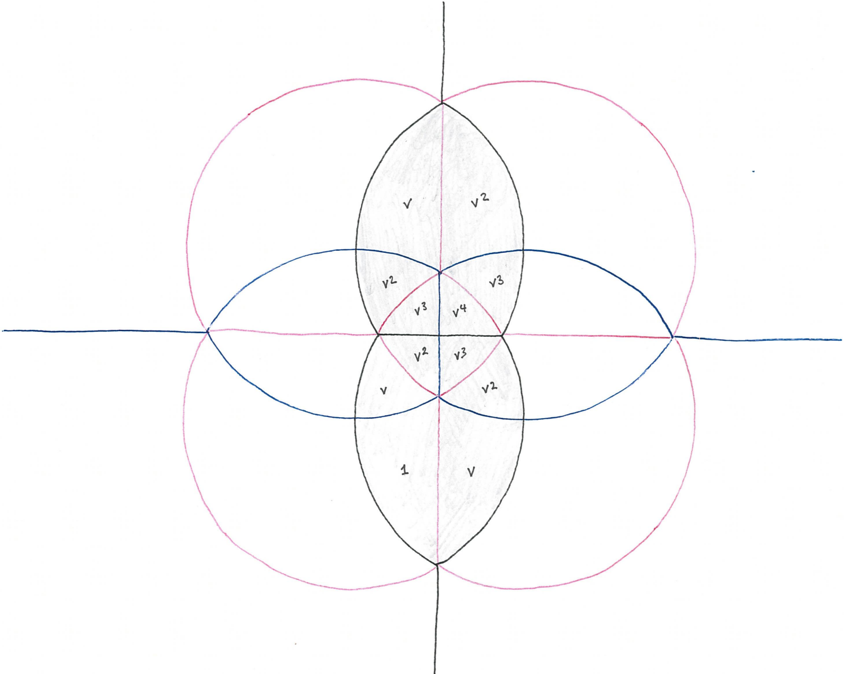

v
v²
1
v

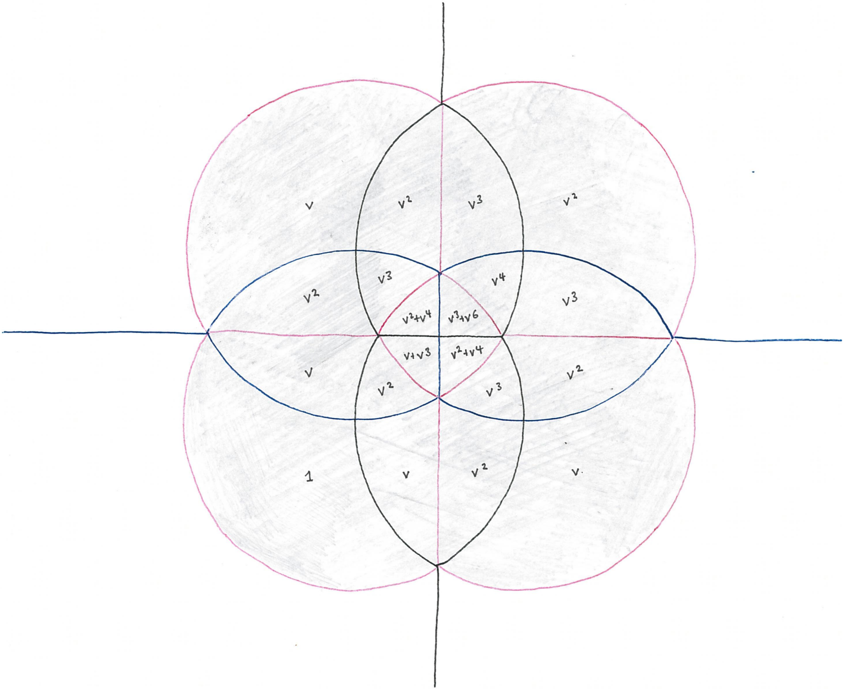


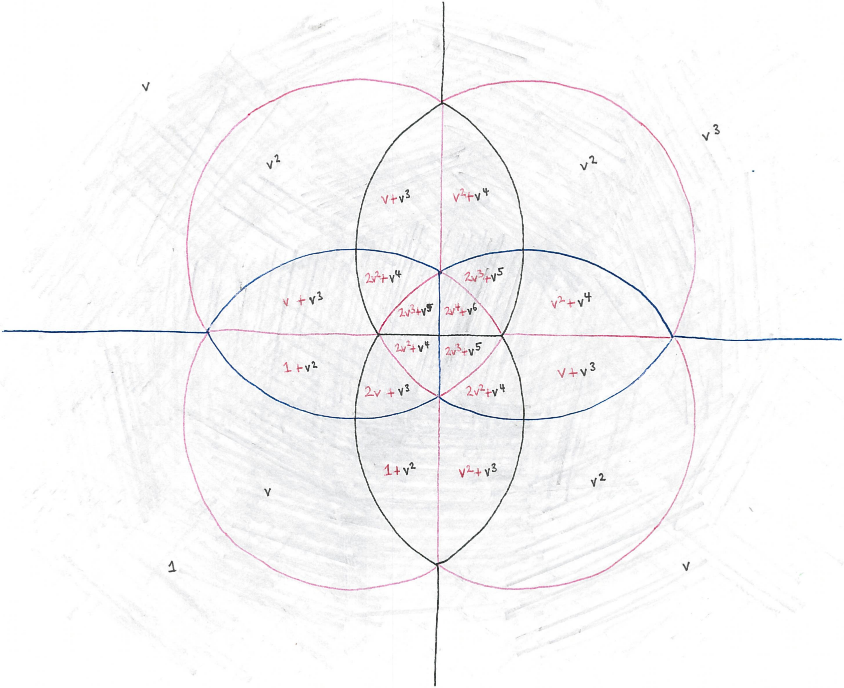
v
v2
v2
v3
v
v2
1
v



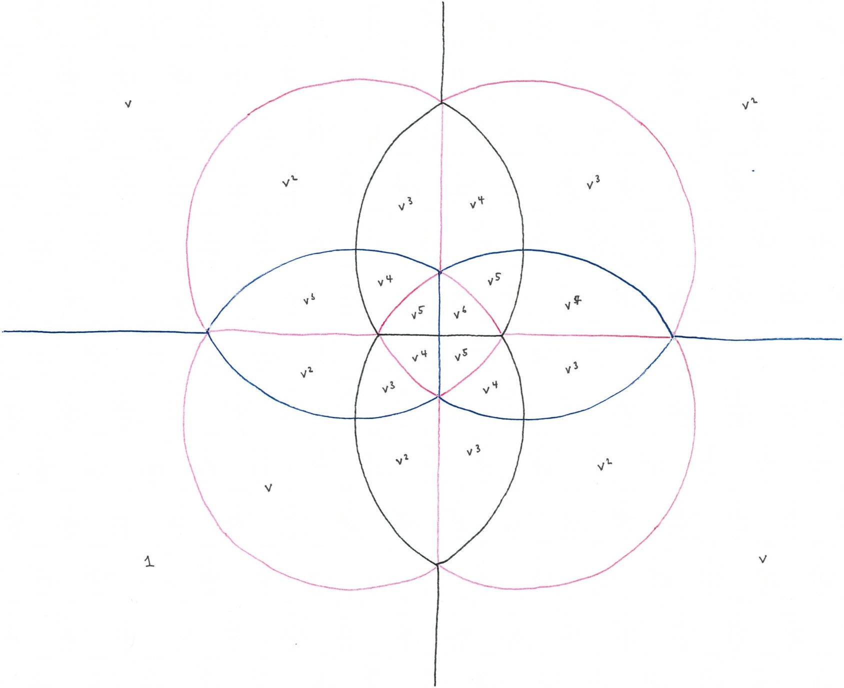
v
v²
v²
v³
v+v³
v²+v⁴
1+v²
v+v³
v
v2
1
v


v
v2
v2
v3
v3
v4
v2
v3
v
v2
1
v


v
v²
v3
v²
v2
v3
v4
v3
v²+v4
v³+v6
v
v+v3
v²+v4
v2
v²
v3
1
v
v²
v.



v
v 3
v 2
v 2
v+v 3
v 2 +v 4
2v 2 +v 4
2v 3 +v 5
v +v 3
2v 3 +v 5
2v 4 +v 6
v 2 +v 4
2v 2 +v 4
2v 3 +v 5
1 +v 2
v+v 3
2v +v 3
2v 2 +v 4
1+v 2
v 2 +v 3
v
v 2
1
v


v
v^2
v^2
v^3
v^4
v^3
v^4
v^5
v^5
v^5
v^6
v^4
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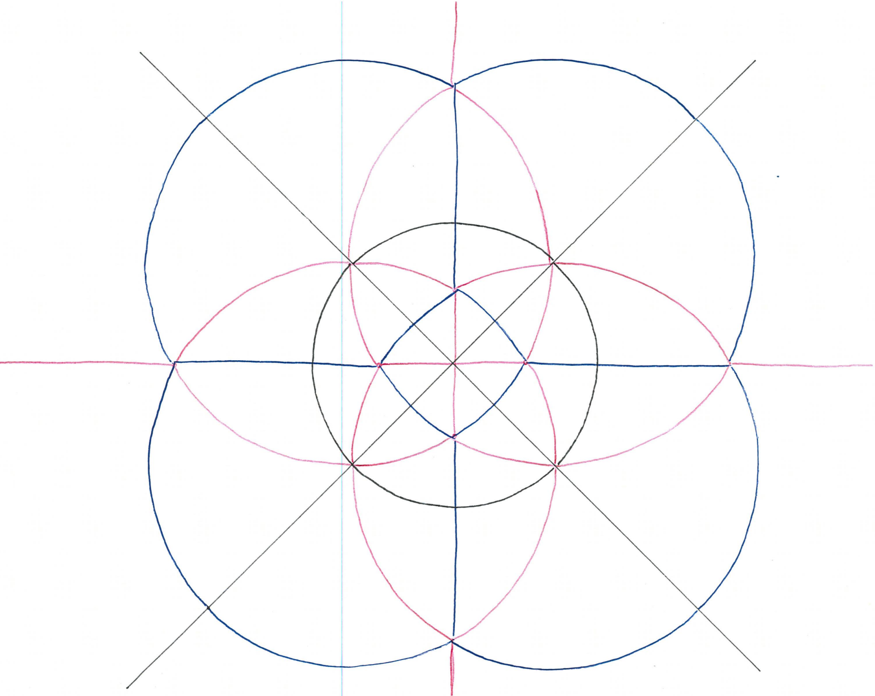
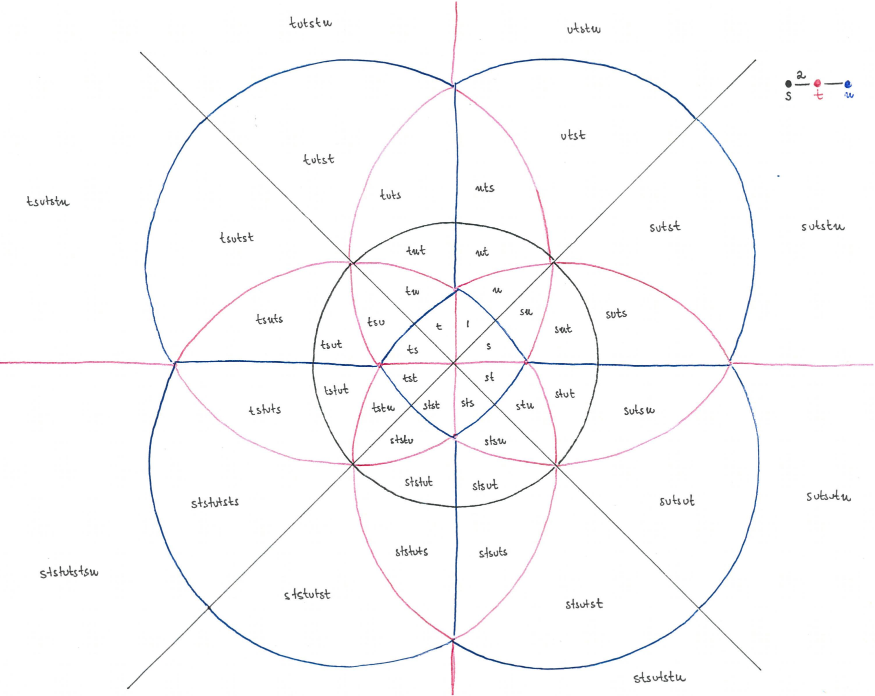
v^3
v^4
v^2
v^3
v^2
v
1
v





2
s
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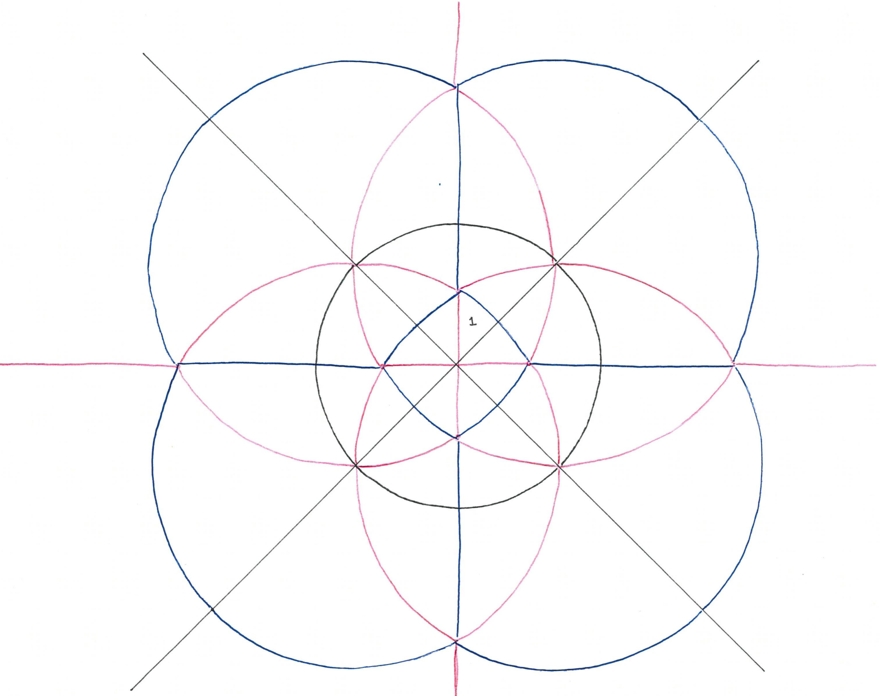
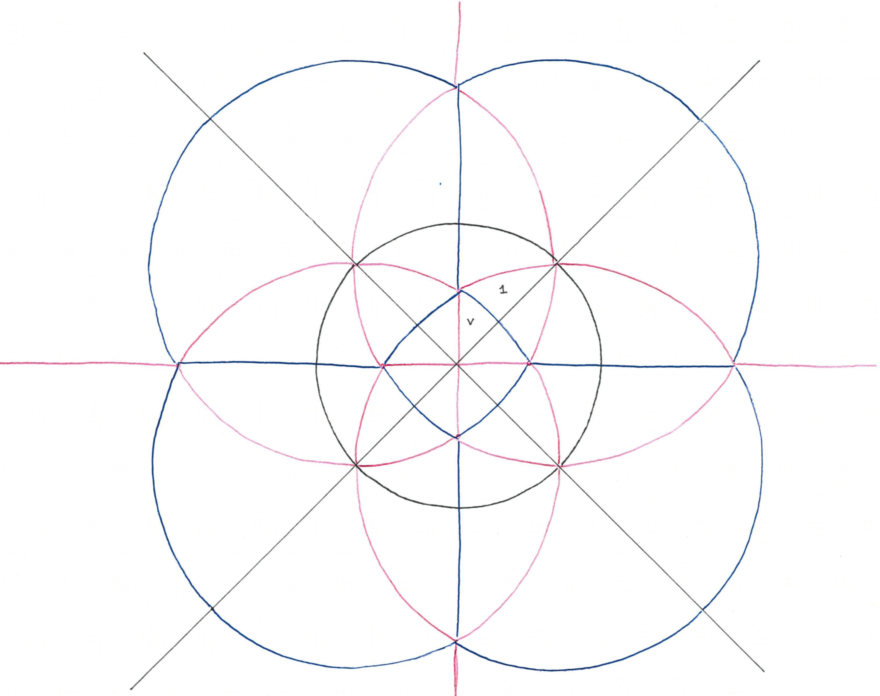
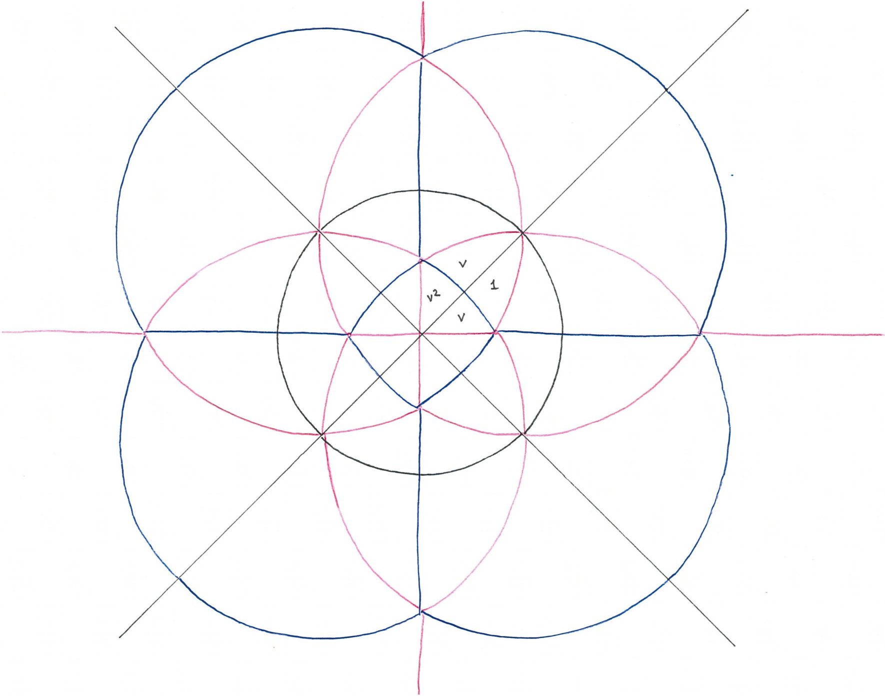
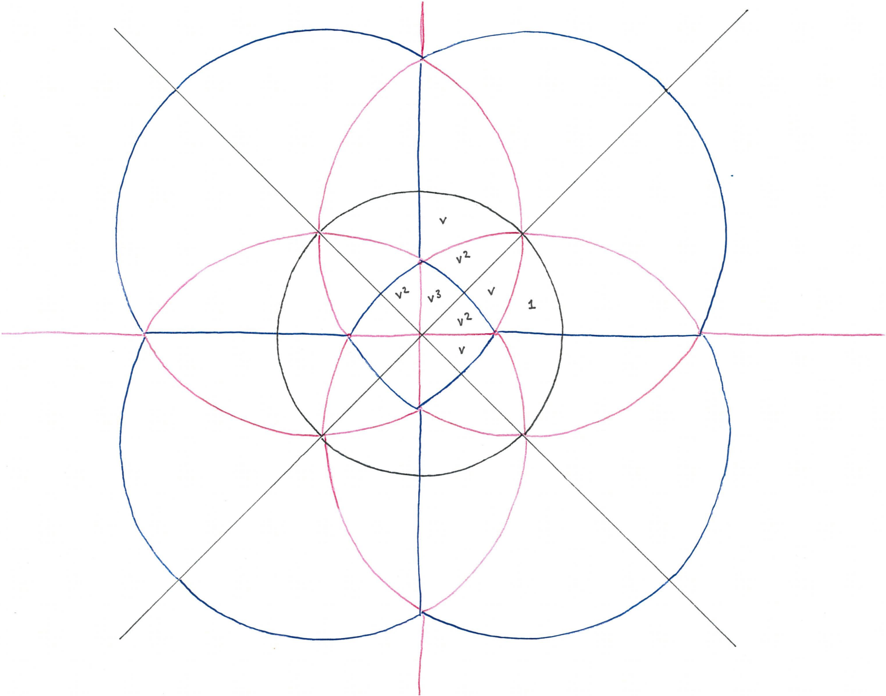
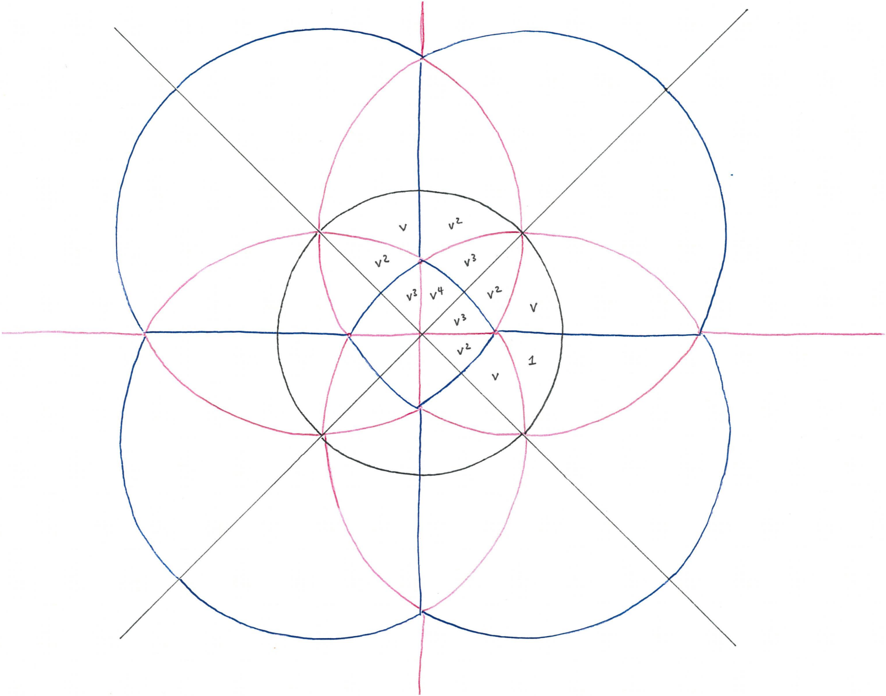

1


1
v

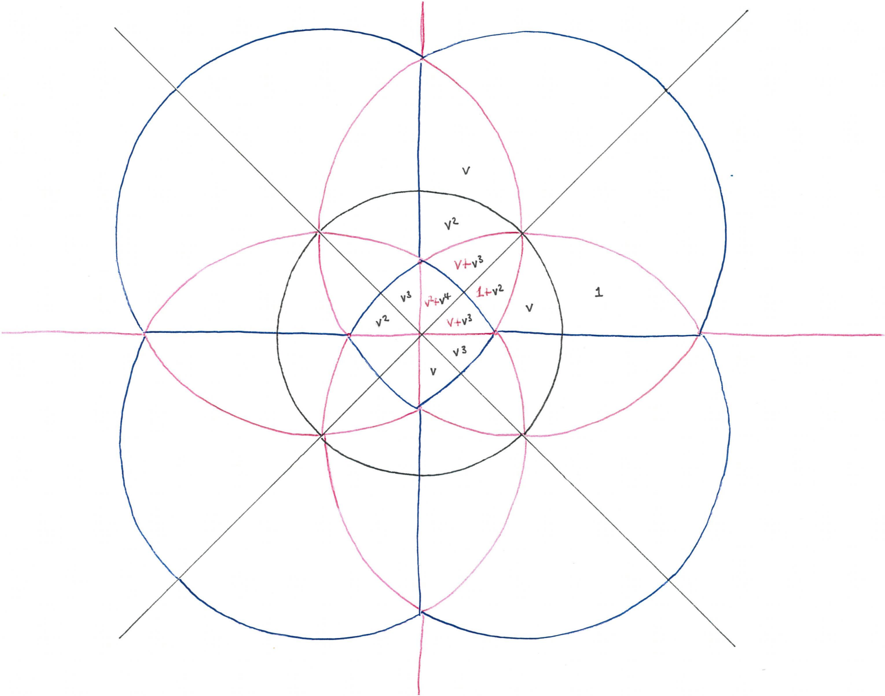

v
v^2
1
v

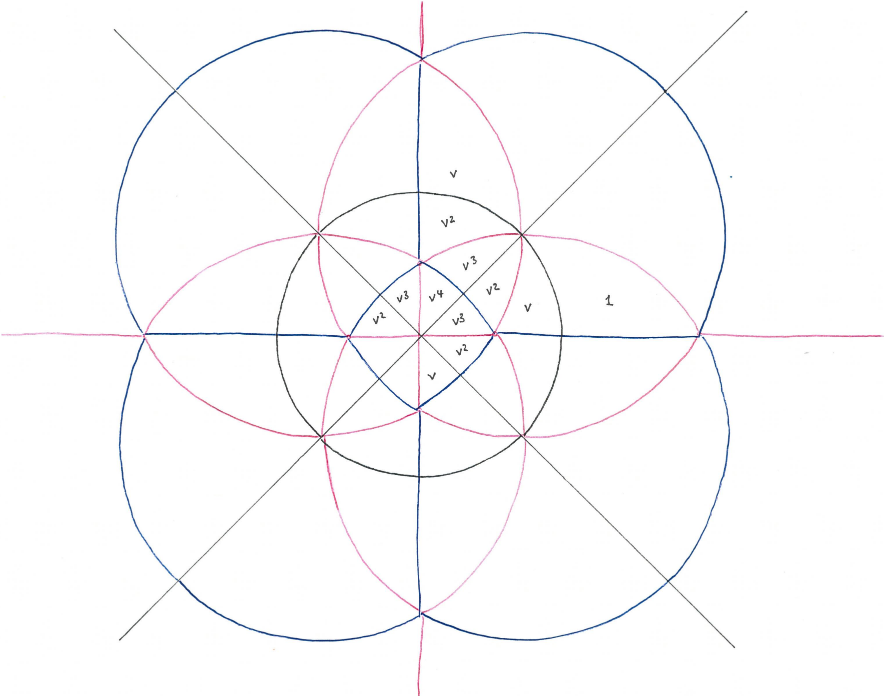


v
v2
v2
v3
v
1
v2
v

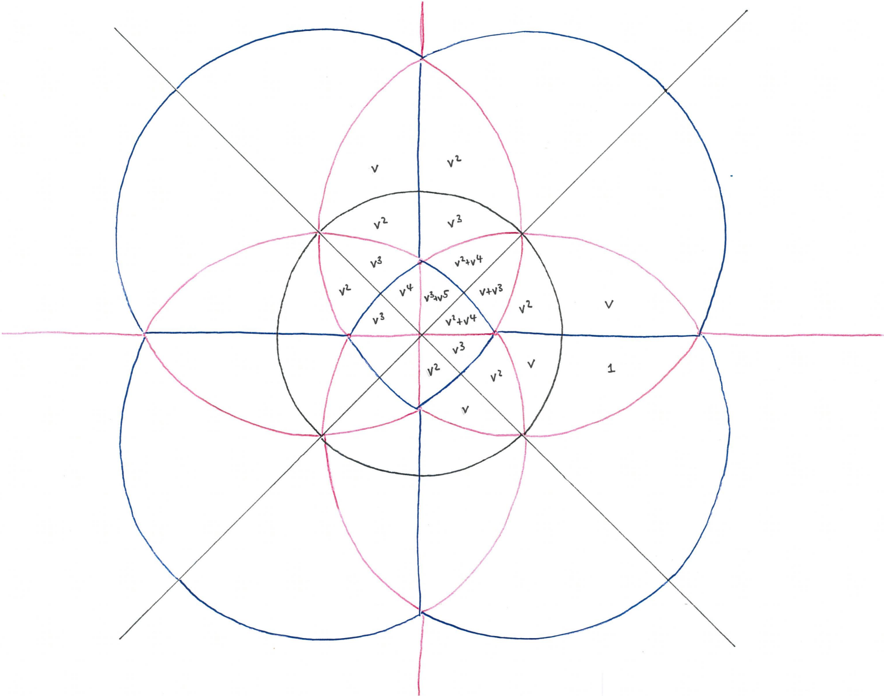


v
v2
v2
v3
v3
v4
v2
v
v3
v2
1
v

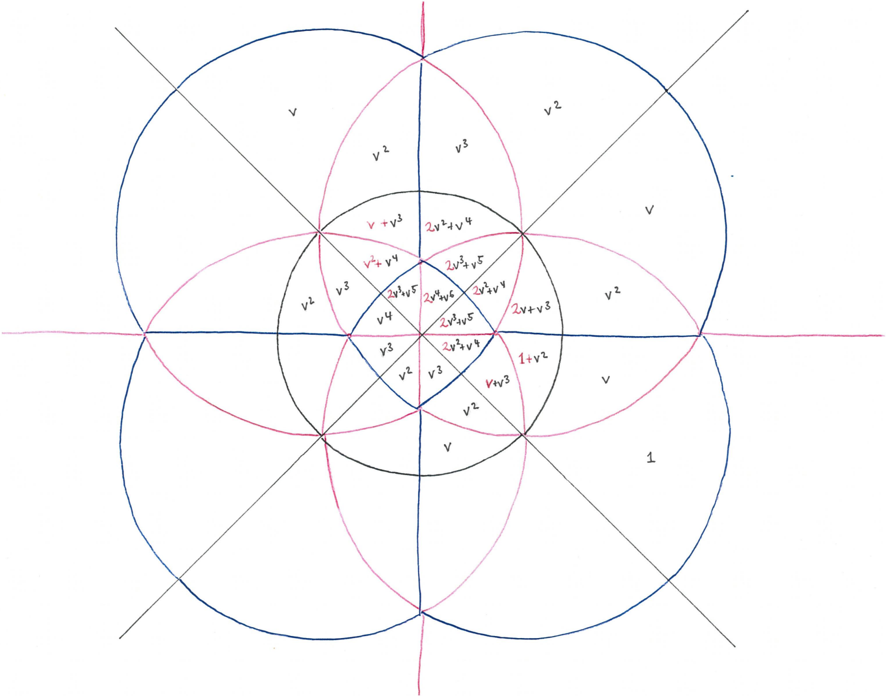

v
v^2
$v+v^3$
$1+v^2$
1
v
v^3
v^2+v^4
v^2
$v+v^3$
v^3
v

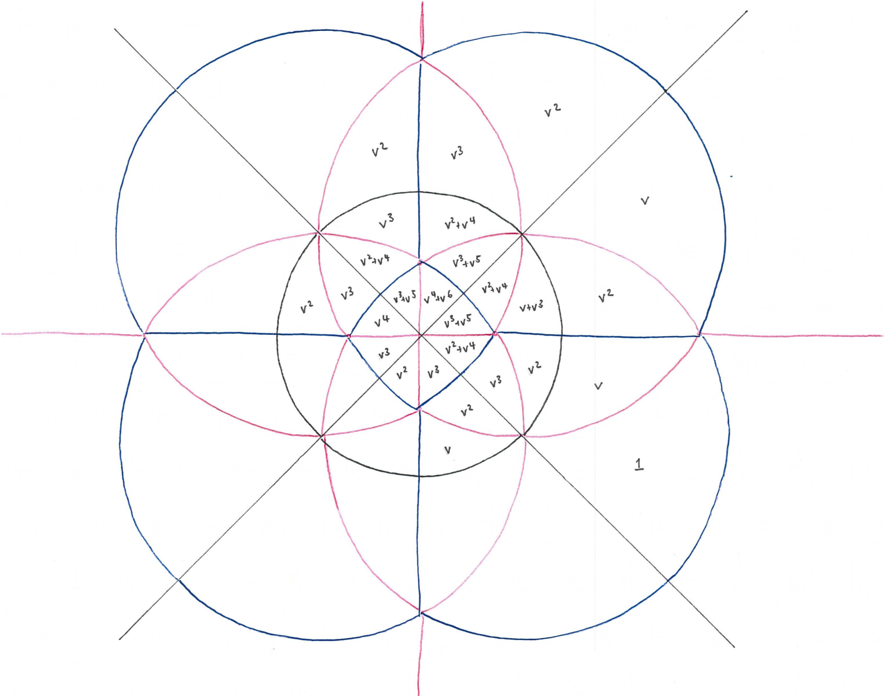


v
v2
v3
v3
v4
v2
v2
v
1
v3
v2
v



v
v2
v2
v3
v3
v2+v4
v2
v4
v3+v5
v+v3
v3
v2+v4
v2
v
v3
v
v2
v2
1
v


v
v^2
v^2
v^3
v
$v+v^3$
$2v^2+v^4$
v^2+v^4
$2v^3+v^5$
v^2
v^3
$2v^3+v^5$
$2v^4+v^6$
$2v^2+v^4$
$2v+v^3$
v^2
v^4
$2v^3+v^5$
v^3
$2v^2+v^4$
$1+v^2$
v^2
v^3
$v+v^3$
v
v^2
v
1


v2
v3
v2
v
v3
v2+v4
v2+v4
v3+v5
v2
v3
v3+v5
v4+v6
v2+v4
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v2+v4
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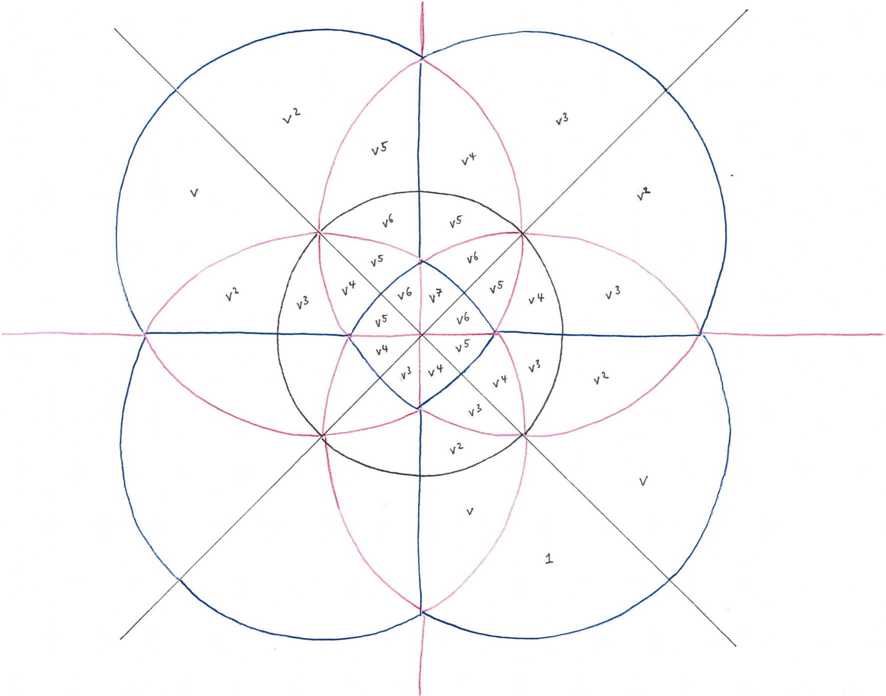
v3
v3
v2
v
v2
v
1


v2
v3
v5
v4
v
v2
v6
v5
v5
v6
v4
v6
v7
v5
v4
v3
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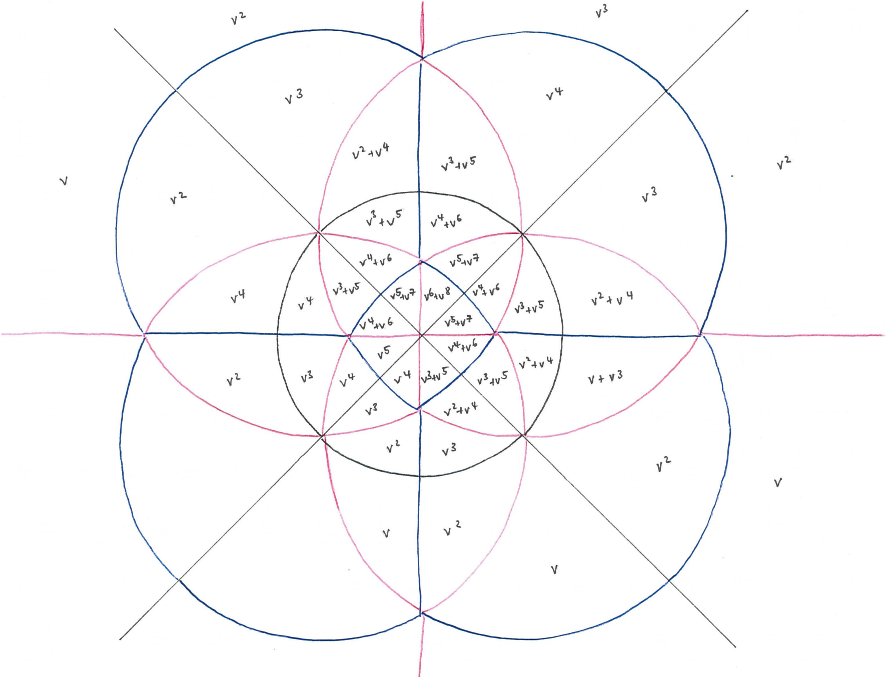
v3
v2
v
v
1


v^2
v^3
$v 3$
v^4
v^2+v^4
v^3+v^5
v
v^2
v^3
v^2
v^3+v^5
v^4+v^6
v^4+v^6
v^5+v^7
$v4$
$v4$
v^3+v^5
v^5+v^7
v^6+v^8
v^4+v^6
v^3+v^5
$v^2+v 4$
$v^4+v 6$
v^5+v^7
v^2
$v3$
$v5$
v^4+v^6
$v^2+v 4$
$v + v3$
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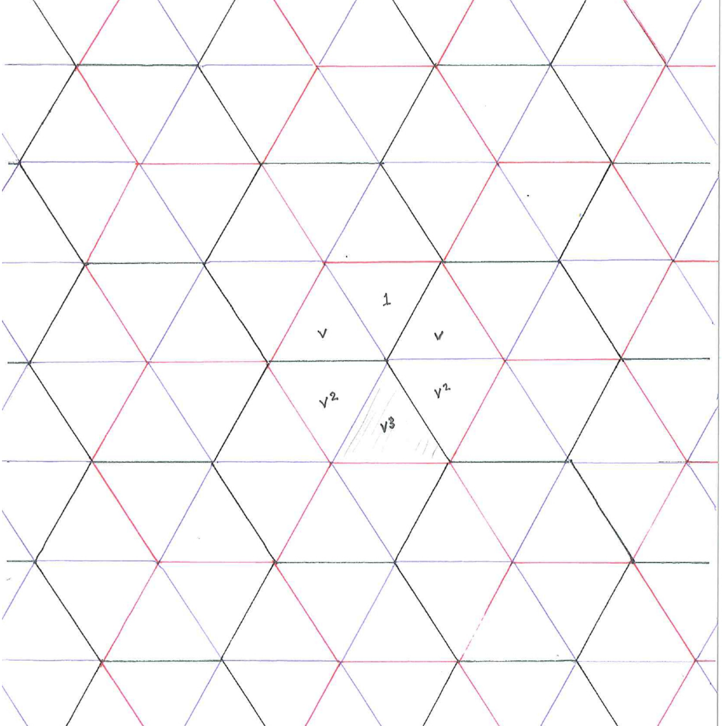
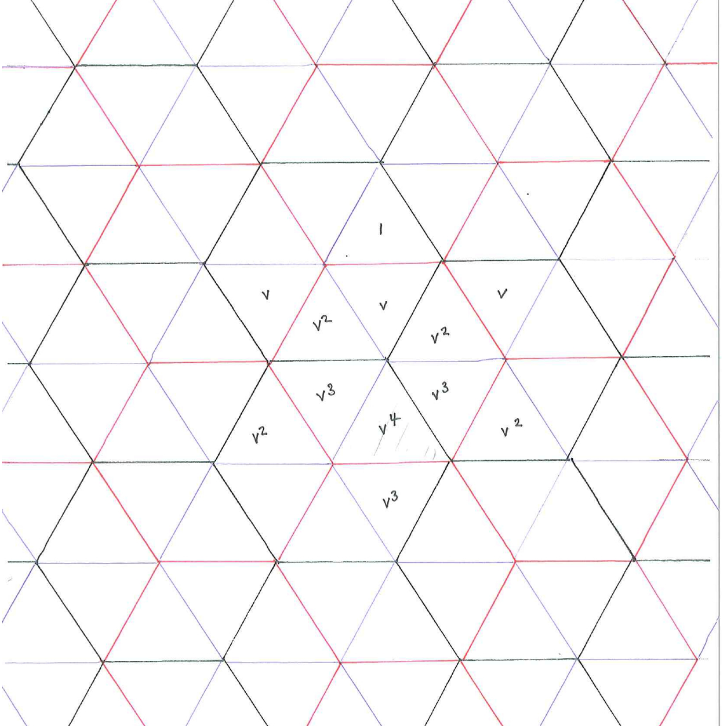
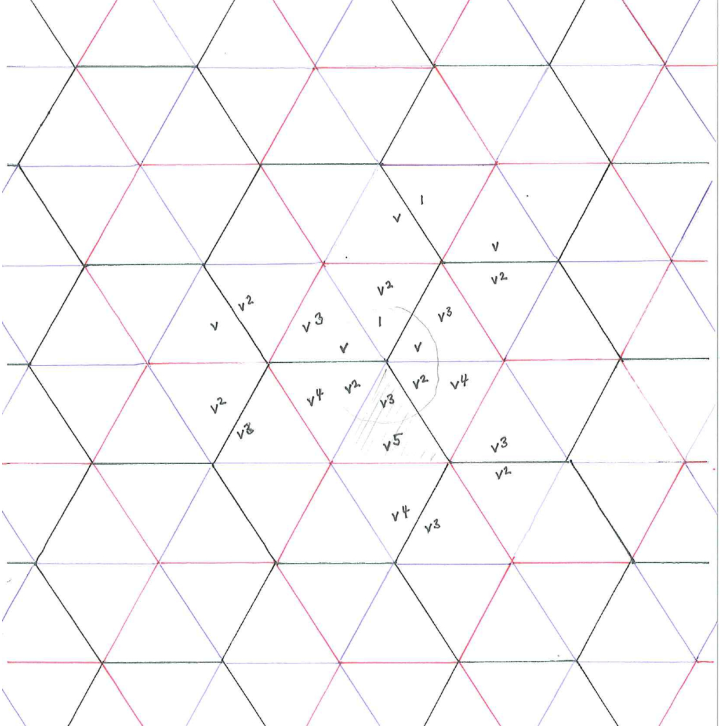
v
v^2
v
v


1
v
v
v2
v2
v3


1
v
v
v
v^2
v^2
v^3
v^3
v^4
v^2
v^2
v^3


v 1
v
v2
v2
v2
v
v3
v3
1
v
v
v2
v2
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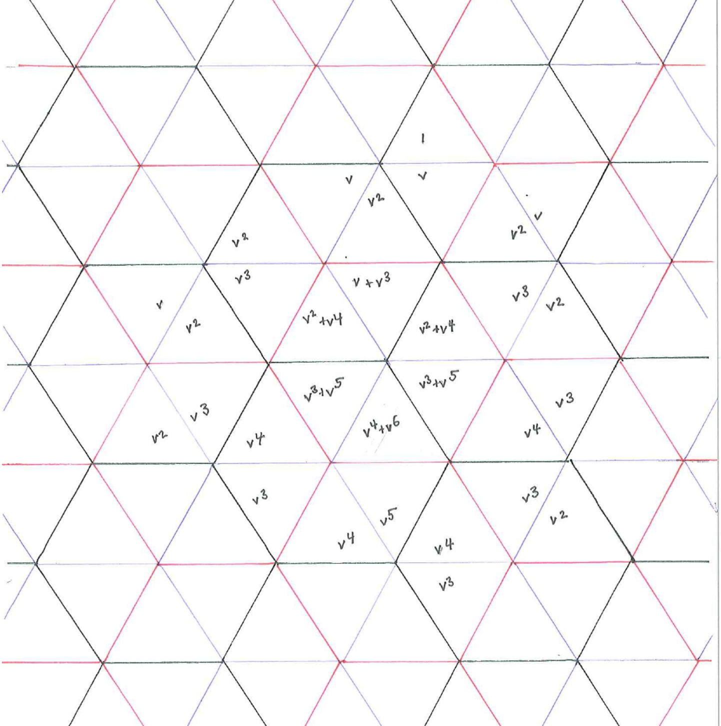
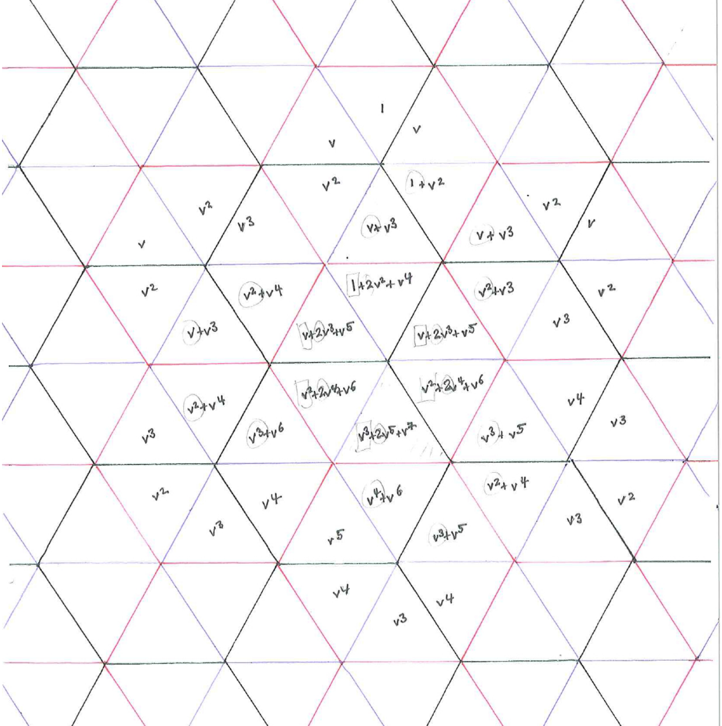
v3
v2
v4
v3


1
v
v
v2
v2
v2
v
v3
v + v3
v
v2
v2+v4
v2+v4
v3
v2
v3+v5
v3+v5
v2
v3
v4
v4+v6
v4
v3
v3
v5
v4
v4
v3
v3
v2


1
v
v
v2
1+v2
v2
v3
v+v3
v+v3
v2
v
v
v2
v2+v4
1+2v2+v4
v2+v3
v2
v+v3
v+2v3+v5
v+2v3+v5
v3
v2+v4
v2+2v4+v6
v2+2v4+v6
v4
v3
v3+v6
v3+2v5+v7
v3+v5
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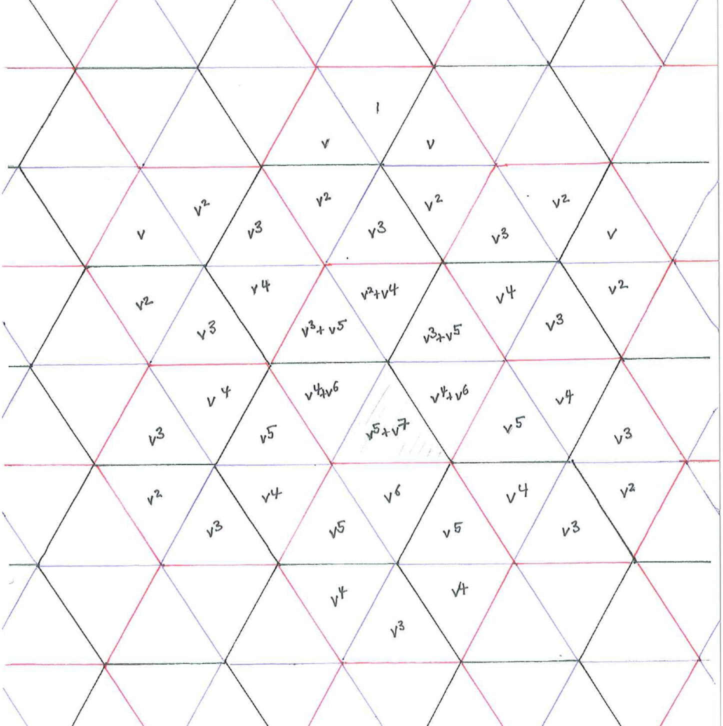
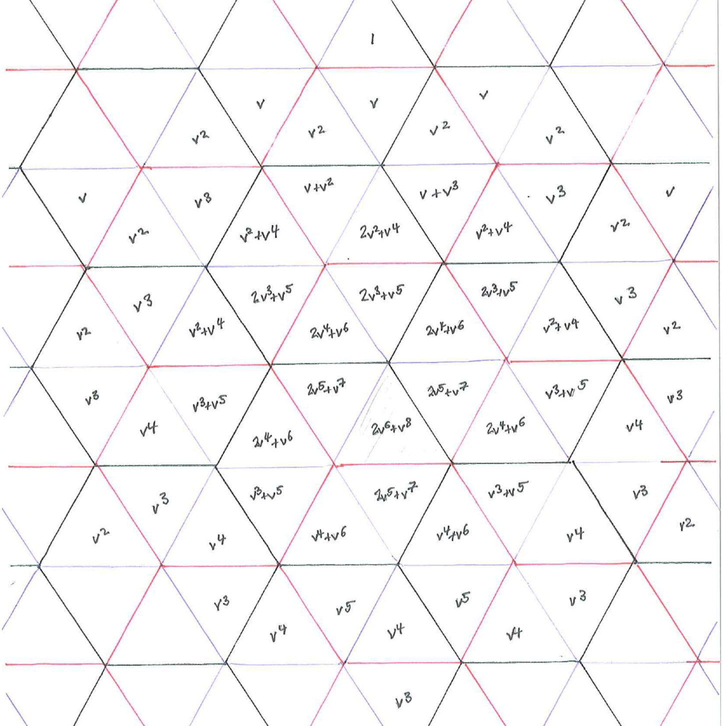
v3
v4
v3
v4


1
v v
v^2 v^2 v^2 v^2
v v^3 v^3 v^3 v
v^2 v^4 v^2+v^4 v^4 v^2
v^3 v^3+v^5 v^3+v^5 v^3
v^4 v^4+v^6 v^4+v^6 v^4
v^3 v^5 v^5+v^7 v^5 v^3
v^2 v^4 v^6 v^4 v^2
v^3 v^5 v^5 v^3
v^4 v^4
v^3


1
v v v
v^2 v^2 v^2 v^2
v v^3 $v+v^2$ $v+v^3$ v^3 v
v^2 v^2+v^4 $2v^2+v^4$ v^2+v^4 v^2
v^3 $2v^3+v^5$ $2v^3+v^5$ $2v^3+v^5$ v^3
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Established for crystallographic W by Kazhdan and Lusztig in 1980, using Deligne's proof of the Weil conjectures.

Crystallographic: $m_{st} \in \{2, 3, 4, 6, \infty\}$.

Why are Kazhdan-Lusztig polynomials hard?

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Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geqslant 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

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Roughly: all positive polynomials are Kazhdan-Lusztig polynomials!

The most complicated Kazhdan-Lusztig-Vogan polynomial computed by the *Atlas of Lie groups and Representations* project:

$$
\begin{aligned}
152q^{22} &+ 3\,472q^{21} + 38\,791q^{20} + 293\,021q^{19} + 1\,370\,892q^{18} + \\
&+ 4\,067\,059q^{17} + 7\,964\,012q^{16} + 11\,159\,003q^{15} + \\
&+ 11\,808\,808q^{14} + 9\,859\,915q^{13} + 6\,778\,956q^{12} + \\
&+ 3\,964\,369q^{11} + 2\,015\,441q^{10} + 906\,567q^{9} + \\
&+ 363\,611q^{8} + 129\,820q^{7} + 41\,239q^{6} + \\
&+ 11\,426q^{5} + 2\,677q^{4} + 492q^{3} + 61q^{2} + 3q
\end{aligned}
$$

(This polynomial is associated to the reflection group of type E_8. See `www.liegroups.org`.)

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Kazhdan-Lusztig conjecture (1979):

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(A major generalisation of the Weyl character formula.)

Implications for representations of real Lie groups.

The Kazhdan-Lusztig conjecture was proved 1981 by Beilinson-Bernstein and Brylinski-Kashiwara using every trick in the book: algebraic differential equations "D-modules"; the Riemann-Hilbert correspondence (monodromy of differential equations); the theory of perverse sheaves (algebraic topology of singular varieties); Deligne's theory of weights (arithmetic geometry):

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"The amazing feature of the proof is that it does not try to solve the problem but just keeps translating it in languages of different areas of mathematics (further and further away from the original problem) until it runs into Deligne's method of weight filtrations which is capable to solve it.

So have a seat; it is going to be a long journey."

– Joseph Bernstein.

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Using results of Soergel we obtain an algebraic proof of the Kazhdan-Lusztig conjecture, as well as algebraic proofs of many of the results mentioned on the previous slide.

Our goal is to understand the Kazhdan-Lusztig positivity conjectures:

$$\underline{H}_x = \sum h_{y,x} H_y \quad h_{y,x} \in \mathbb{Z}_{\geqslant 0}[v]$$
$$\underline{H}_x \underline{H}_y = \sum \mu^z_{x,y} \underline{H}_z \quad \mu^z_{x,y} \in \mathbb{Z}_{\geqslant 0}[v^{\pm 1}].$$

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A basic principle in combinatorics to show that a number is positive is to show that it is the cardinality of a set or the dimension of a vector space.

This is a baby example of *categorification*. One upgrades a number to an object in a category (in this example a set or vector space).

Given an abelian category $\mathcal{A}$ its *Grothendieck group* is

$$K_0(\mathcal{A}) = \bigoplus_{M \in \mathcal{A}} [M] / \left(\begin{array}{c} [M] = [M'] + [M''] \\ \text{for all short exact sequences} \\ 0 \to M' \to M \to M'' \to 0 \end{array} \right).$$

Given an additive category $\mathcal{B}$ its *split Grothendieck group* is

$$K_0^{\text{split}}(\mathcal{B}) = \bigoplus_{B \in \mathcal{B}} [B] / \left(\begin{array}{c} [B] = [B'] + [B''] \\ \text{whenever } B \cong B' \oplus B'' \end{array} \right).$$

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The passage from a category to its (split) Grothendieck group is the process of *decategorification*. Finding **interesting** inverses to this procedure is the process of *categorification*.

Categorifying the Hecke algebra:

For simplicity assume that $W \subset O(V)$ is a finite reflection group. Set

$$R = \text{polynomial functions on } V$$

graded such that $V^* \subset R$ has degree 2. If we choose a basis $X_1, \ldots, X_m$ of V^* then R is simply the polynomial ring $\mathbb{R}[X_1, \ldots, X_m]$. Because W acts on V, it also acts on R. For any simple reflection $s \in S$ consider $R^s \subset R$ the subalgebra of s-invariants (also a polynomial ring).

Let $R - \mathrm{Bim}$ denote the monoidal category of graded R-bimodules: $MM' := M \otimes_R M'$.

We denote by (1) the grading shift operator: $M(1)^i = M^{i+1}$.

For $s \in S$ let $B_s := R \otimes_{R^s} R$. Consider

$$\mathcal{B} = \begin{array}{c} \text{full Karoubian subcategory of } R - \mathrm{Bim} \\ \text{generated by } B_s(m) \text{ for all } s \in S,\ m \in \mathbb{Z}. \end{array}$$

In other words, the objects $\mathcal{B}$ are the graded R-bimodule direct summands of bimodules of the form

$$B_s B_t \dots B_u = R \otimes_{R^s} R \otimes_{R^u} R \otimes \dots \otimes_{R^u} R(m)$$

for arbitrary sequences $st \dots u$ and $m \in \mathbb{Z}$.

Let $K_0^{\text{split}}(\mathcal{B})$ denote the split Grothendieck group of $\mathcal{B}$. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ via $v[B] := B(1)$ and $[B'][B''] = [B'B'']$.

Soergel's categorification theorem (2005):

The split Grothendieck group of $\mathcal{B}$ is isomorphic to the Hecke algebra:

$$\mathrm{ch} : K_0^{\mathrm{split}}(\mathcal{B}) \xrightarrow{\sim} H$$

First example: $W = S_2 = \{id, s\}$ acting on $\mathbb{R}$.

Then $R = \mathbb{R}[X]$ with $sX = -X$ and $R^s = \mathbb{R}[X^2]$.

Any polynomial is the sum of an even and an odd polynomial. Hence we have an isomorphism of $\mathbb{R}[X^2]$-bimodules:

$$R = \mathbb{R}[X] = \mathbb{R}[X^2] \oplus X\mathbb{R}[X^2] = R^s \oplus R^s(-2).$$

Hence

$$B_s B_s = R \otimes_{R^s} R \otimes_{R^s} R(2) = R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) = \\ = (R \otimes_{R^s} R)(2) \oplus (R \otimes_{R^s} R) = B_s(1) \oplus B_s(-1).$$

This categorifies the relation $\underline{H}_s^2 = v^{-1}\underline{H}_s + v\underline{H}_s$ and shows that (up to isomorphism and shifts) there are only two indecomposable Soergel bimodules: $B_{id} = R$ and B_s.

Second example: $W = S_3$ acting on $R = \mathbb{R}[X_1, X_2, X_3]$ by permutations of the variables. Set $s = (1,2)$ and $t = (2,3)$. Then $R^s = \mathbb{R}[X_1 + X_2, X_1X_2, X_3]$ etc. and one can check (with some effort) the following facts:

1. B_s, B_t, B_sB_t and B_tB_s are cyclic (hence indecomposable) R-bimodules.
2. one has isomorphisms of graded R-bimodules

$$B_sB_tB_s = B_{sts} \oplus B_s \qquad B_tB_sB_t = B_{sts} \oplus B_t$$

   where $B_{sts} = R \otimes_{R^{s,t}} R(3)$.
3. $B_{sts}B_s \cong B_{sts}(1) \oplus B_{sts}(-1) \cong B_{sts}B_t$.

All these isomorphisms categorify facts in the Hecke algebra (e.g. $\underline{H}_s\underline{H}_t = \underline{H}_{st}$, $\underline{H}_{sts}\underline{H}_s = (v + v^{-1})\underline{H}_{sts}$ etc.)

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They also show that the set of indecomposable Soergel bimodules (up to isomorphism and shifts) consist of

$$\{R = B_{id}, B_s, B_t, B_{st}, B_{ts}, B_{sts}\}.$$

To prove his categorification theorem, Soergel establishes the existence and uniqueness of indecomposable bimodules B_x whose classes give a basis for $K_0^{\text{split}}(\mathcal{B})$ and conjectures:

Soergel's conjecture:

$$\mathrm{ch}(B_x) = \underline{H}_x$$

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$\Rightarrow$Kazhdan-Lusztig positivity conjectures.

Soergel also explained how his category of bimodules sees everything about category $\mathcal{O}$.

In particular his conjecture implies the Kazhdan-Lusztig conjecture on characters of simple highest weight modules.

An important role in our inductive proof of Soergel's conjecture is played by certain much stronger statements about the bimodules B_x.

Set $\overline{B_x} := B_x \otimes_R \mathbb{R}$. This is a finite dimensional graded vector with left R action on the left. It is equipped with a non-degenerate symmetric "intersection form" $\langle -, - \rangle$.

Example: if $w_0 \in W$ denotes the longest element then $B_{w_0} = R \otimes_{R^W} R(\ell(w_0))$ and hence

$$\overline{B_{w_0}} = R/(R^W)^+(\ell(w_0))$$

is the "coinvariant algebra": $(R^W)^+$ denotes the ideal of R generated by W-invariant polynomials of positive degree.

We show that $\overline{B_x}$ "looks like the cohomology of a smooth projective variety".

For any $\rho \in V^*$ in the interior of the fundamental alcove we have:

i) (Hard-Lefschetz theorem) left multiplication by ρ^i gives an isomorphism
$$(\overline{B_x})^{-i} \to (\overline{B_x})^i$$

1. (Hodge-Riemann bilinear relations) The restriction of the form $(\alpha, \beta) := \langle \alpha, \rho^i \beta \rangle$ to the kernel of ρ^{i+1} in $(\overline{B_x})^{-i}$ is definite.

What does this say for the coinvariant algebra?

If W is a Weyl group then $R/(R^W)^+$ is isomorphic to the cohomology ring of the flag variety. Flag varieties are smooth projective varieties and these properties follow from classical Hodge theory.

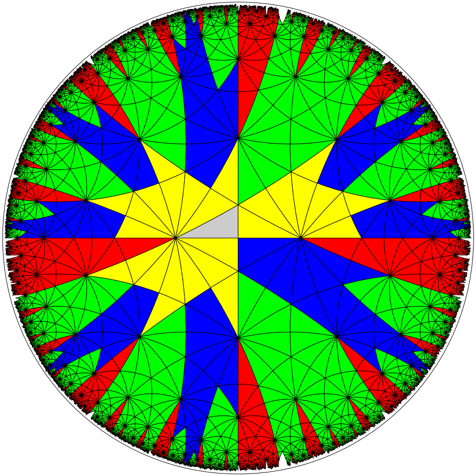
If W is not a Weyl group (for example the symmetries of the icosahedron), there is no algebraic variety with the coinvariant algebra as cohomology.

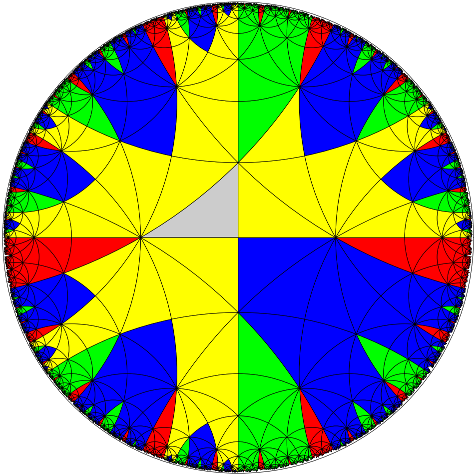
Here even for H_3 the Hodge-Riemann bilinear relations are new, and the only proofs of hard Lefschetz were computer assisted.

To any element of any Coxeter group W one has a space which looks like the cohomology of a smooth projective variety!

I will finish with two questions:

i) Is there any geometric interpretation of these spaces? (One can ask a similar question for the intersection cohomology of non-rational polytopes.)
ii) What does Kazhdan-Lusztig theory mean in the non-crystallographic case?





For more images of two-sided cells in hyperbolic groups see Paul Gunnell's web page.