Koszul Duality and Geometric Satake for $SL_2(\mathbb{R})$

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Motivation

- $G = PSL_2(\mathbb{C})$, $\check{\mathfrak{g}} := \mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C}$ and $K = SO_2(\mathbb{R})$

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Question

Is there a nice (geometric) desciption of "?"

Outline

Some Notation

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- For any infinitisimal character $n \in \mathbb{N}_0$

$$\mathscr{HC}_n = \{M \in \mathscr{HC} | \ (n(z) - z)^k M = 0 \text{ for } k \gg 0 \text{ and } \forall z \in Z(U(\check{\mathfrak{g}}))\}$$

and *the category of Harish-Chandra Modules with integral infinitisimal character*

$$\mathscr{HC} = \oplus_{n \in \mathbb{N}_0} \mathscr{HC}_n$$

is stable under tensoring with finite dimensional $SL_2(\mathbb{C})$ representations.

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- Let $D^{ss}_G(X)$ full additive subcategory of semisimple objects of $D^b_G(X)$. (An object in $D^b_G(X)$ is called semisimple if it is isomorphic to finte direct sum of shifted simple objects of $Perv_G(X)$)

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Theorem (Geometric Satake Isomorphism, Mirkovic-Vilonen)

There exists an equivalence of tensor categories

$$\mathcal{S} : (Perv_{G(\mathcal{O})}(Gr), *) \to (Rep_{\mathbb{C}}(SL_2(\mathbb{C})), \otimes)$$

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Problem

Unfortunately $G(\mathcal{K})$ has no obvious action on X

Lemma

Let $\mathscr{X}^n := G(\mathcal{O})t^n G(\mathcal{O}) \times_{G(\mathcal{O})} Y$. *There exists an equivalence of categories*

$$D^b_{G(\mathcal{O})}(\mathscr{X}^n) \cong D^b_G(X(n))$$

preserving the perverse t-structure.

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but still $G(\mathcal{K})$ does not operate continously on $\widetilde{\mathscr{X}}$

Now we will change the topology of $\widetilde{\mathscr{X}}$ a little, such that $G(\mathcal{K})$ acts continiously as follows:

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Proposition

There exists a bi-functor

$$\_ * \_ : D^b_{G(\mathcal{O})}(Gr) \times D^b_{G(\mathcal{O})}(\mathscr{X}) \to D^b_{G(\mathcal{O})}(\mathscr{X})$$

generalizing the convolution bi-functor on the Affine Grassmannian.

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Let $F \in Perv_{G(\mathcal{O})}(Gr)$. Geometric Tensorfunctor $F \tilde{*} \_$ is defined as follows:

$$\begin{tikzcd} D^b_{G(\mathcal{O})}(\widetilde{\mathscr{X}}) \arrow[r, "Rj_!"] \arrow[d, "F \tilde{*} \_"] & D^b_{G(\mathcal{O})}(\mathscr{X}) \arrow[d, "F * \_"] \\ D^b_{G(\mathcal{O})}(\widetilde{\mathscr{X}}) & D^b_{G(\mathcal{O})}(\mathscr{X}) \arrow[l, "j^*"'] \end{tikzcd}$$

Main Theorem

Recall: By Koszul Duality $D^{ss}_{G(\mathcal{O})}(\widetilde{\mathscr{X}})$ is a graded version of the "projective objects" of $\mathscr{HC}$.

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Theorem

Geometric Tensoration is the Koszul-Dual of Tensoration with finite dimensional representations.
If F is a finite dimensional rational $SL_2(\mathbb{C})$-representation and $\mathcal{S}(F)$ it's Satake equivalent, then the following diagram commutes.

$$\begin{tikzcd}
P\mathscr{H}\mathscr{C} \arrow[d, "F\otimes\_"] & D^{ss}_G(\widetilde{\mathscr{X}}) \arrow[l, "for"'] \arrow[d, "\mathcal{S}(F)\tilde{*}\_"] \\
P\mathscr{H}\mathscr{C} & D^{ss}_G(\widetilde{\mathscr{X}}) \arrow[l, "for"']
\end{tikzcd}$$

Corollary

For each finite dimensional representation F *of* $SL_2(\mathbb{C})$ *there exists a graded lift* $F\widetilde{\otimes}\_ : \mathscr{HC}^{\mathbb{Z}} \to \mathscr{HC}^{\mathbb{Z}}$ *of the functor* $F \otimes \_ : \mathscr{HC} \to \mathscr{HC}$*,*

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- *For the trivial representation $\mathbb{C}$ we have an isomorphism of functors*

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- *For the trivial representation* $\mathbb{C}$ *we have an isomorphism of functors*
$$\mathbb{C}\widetilde{\otimes}\_ \cong id$$
- *For* $F_1, F_2 \in Rep(SL_2(\mathbb{C}))$ *we have an isomorphism of functors*
$$(F_1 \otimes F_2)\widetilde{\otimes}\_ \cong F_1\widetilde{\otimes}(F_2\widetilde{\otimes}\_)$$

Some Properties of the Geometric Tensor Functor

Definition (Geometric Translation Functors)

For $k, m, n \in \mathbb{N}_0$ let $L(|n-m|)$ be the unique simple finite dimensional representation with highest weight $|n-m|$ and $j_k : X^k \to X$ the inclusion.

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$$\begin{tikzcd}
D^b_{G(\mathcal{O})}(\mathscr{X}^m) \arrow[r, "Rj_{m!}"] \arrow[d, "\mathscr{T}_{m,n}"] & D^b_{G(\mathcal{O})}(\mathscr{X}) \arrow[d, "\mathcal{S}(L(|n-m|))*\_\_"] \\
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\end{tikzcd}$$

Theorem

For any $\mathcal{F} \in Perv_{G(\mathcal{O})}(Gr)$, there exists an isomorphism of functors

$$\mathcal{F}\tilde{*}\_ \cong \left\{ \begin{matrix} \textit{some finite sum of compositions} \\ \textit{of geometric translation functors} \end{matrix} \right\}$$

non canonically.

Theorem

Let $\pi : G \times_B Y \to Y$ *the quotient map. Then*

$$\mathscr{T}_{m,n} \cong \begin{cases} \pi^*[1] & \textit{if } m = 0,\ n \neq 0 \\ \pi_*[1] & \textit{if } n = 0,\ m \neq 0 \\ \textit{id} & \textit{else} \end{cases}$$

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Corollary

i) *Geometric Tensoration preserves semi-simplicity.*

ii) *Geometric Tensoration is (weakly) associative, this means there exists isomorphism of functors*

$$(\mathcal{F}_1 * \mathcal{F}_2)\tilde{*}\_ \cong \mathcal{F}_1 \tilde{*} (\mathcal{F}_2 \tilde{*}\_)$$

Thank You!

Convolution

Assume $G(\mathcal{K})$ acts continously on X, where $X \in \{Gr, \mathscr{X}\}$. Philisophically the *convolution bi-functor*

$$\_ * \_ : D^b_{G(\mathcal{O})}(Gr) \times D^b_{G(\mathcal{O})}(X) \quad \rightarrow \quad D^b_{G(\mathcal{O})}(X)$$

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is defined as follows: For $\mathcal{F} \in D^b_{G(\mathcal{O})}(Gr)$ define $\mathcal{F} * \_$ by

$$\begin{tikzcd}
D^b_{G(\mathcal{O})}(X) \arrow[r, "\mathcal{F}\boxtimes\_"] \arrow[dddd, "\mathcal{F} * \_"] & D^b_{G(\mathcal{O}) \times G(\mathcal{O})}(Gr \times X) \arrow[d, "p_1^* \times id"] \\
& D^b_{G(\mathcal{O})^3}(G(\mathcal{K}) \times X) \arrow[d, "res_\Delta"] \\
& D^b_{G(\mathcal{O})^2}(G(\mathcal{K}) \times X) \arrow[d, "(q^*)^{-1}"] \\
& \\
D^b_{G(\mathcal{O})}(X) & D^b_{G(\mathcal{O})^2}(G(\mathcal{K}) \times_{G(\mathcal{O})} X) \arrow[l, "m_!"']
\end{tikzcd}$$

Summerschool on Category $\mathcal{O}$

- Freiburg: 6.8-9.8.2013
- http://home.mathematik.uni-freiburg.de/ostraser/cato.html