Reduction theorem for global/local conjectures on blocks of a finite group

Britta Späth

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Topic

Central objects and notations:

G	a finite group
$\mathrm{Irr}(G)$	$=$ {irreducible complex characters of G}
	$\overset{1:1}{\leftrightarrow}$ {isomorphism classes of simple $\mathbb{C}G$-modules}
p	a prime
$\mathbb{F}$	algebraically closed field of characteristic p
$\mathrm{IBr}(G)$	$=$ {irreducible Brauer characters of G}
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Study $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ via subgroups of G!

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It is known:

- Using the classification of finite simple groups, strong statements on groups and their representations can be proven.
- The representations theory of p-solvable groups seems well-understood.
- Quasisimple groups have a rich geometric/combinatorial structure or are accessible via computer calculations.

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Conjecture - McKay

We set $\mathrm{Irr}_{p'}(G) := \{\chi \in \mathrm{Irr}(G) \mid p \nmid \chi(1)\}$.

McKay conjecture (1972)

Let $P \in \mathrm{Syl}_p(G)$ a Sylow p-subgroup.
Then

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P))|.$$

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Example: McKay Conjecture for $G = SL_4(\mathbb{F}_2)$ and $p = 2$

Character Degrees in $\mathrm{Irr}(G)$ (with multiplicities):

$$|\mathrm{Irr}_{2'}(G)| = 8$$

$P \quad = \{$upper unitriangular matrices$\}$ is a Sylow 2-subgroup

$N_G(P) = \{$upper triangular matrices$\}$

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Example: McKay Conjecture for $G = \mathrm{SL}_4(\mathbb{F}_2)$ and $p = 5$

Character Degrees in $\mathrm{Irr}(G)$ (with multiplicities):

$$|\,\mathrm{Irr}_{5'}(G)| = 9$$

A Sylow 5-subgroup P is isomorphic to $\mathbb{Z}/5\mathbb{Z}$ and can be realised as a subgroup of the torus in $\mathrm{SL}_4(\mathbb{F}_{16})$.

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Example: McKay Conjecture for $G = \mathrm{SL}_4(\mathbb{F}_2)$ and $p = 3$

Character Degrees in $\mathrm{Irr}(G)$ (with multiplicities):

$$1, 7, 14, 20, 21, 21, 21, 28, 35, 45, 45, 56, 64, 70$$

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A Sylow 3-subgroup P is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and can be realised as a subgroup of the torus in $\mathrm{SL}_4(\mathbb{F}_4)$.

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Modular representation theory

$(\mathcal{K}, \mathcal{O}, \mathbb{F})$	p-modular system
$\mathcal{O}$	complete discrete evaluation ring, such that
$\mathcal{K} = \mathrm{Frak}(\mathcal{O})$	field of characteristic 0
$\mathbb{F} := \mathcal{O}/\,\mathrm{J}(\mathcal{O})$	a field with $\mathrm{char}(\mathbb{F}) = p$

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$$\mathcal{K} \hookleftarrow \mathcal{O} \twoheadrightarrow \mathbb{F}$$

Blocks of finite groups

- $\mathbb{F}G$ not semisimple, whenever $p \mid |G|$

$$\mathbb{F}G = B_1 \oplus \cdots \oplus B_s$$

- indecomposable $\mathbb{F}[G \times G]$-modules called p-blocks of G
- $\mathrm{Bl}(G) = \{B_1, \ldots, B_s\}$

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$$\mathcal{K}G \hookleftarrow \mathcal{O}G \twoheadrightarrow \mathbb{F}G$$

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Defect group and Brauer correspondent of a block

Let B be a p-block of G.

Local data of B:

- defect group D of B, p-subgroup of G
- the p-block b of $N_G(D)$ with defect group D, such that $b^G = B$. This block is called Brauer correspondent of B.

Definition :

Let $\chi \in \mathrm{Irr}(B)$. The **height of** χ, denoted by $\mathrm{ht}(\chi)$ is defined by

$$p^{\mathrm{ht}(\chi)} = \frac{\chi(1)_p |D|}{|G|_p}.$$

- Note $\mathrm{ht}(\chi) \geq 0$.
- Always: $\mathrm{Irr}_0(B) := \{\chi \in \mathrm{Irr}(B) \mid \mathrm{ht}(\chi) = 0\} \neq \emptyset$

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Blocks with p'-characters

Lemma

Let P be a Sylow p-subgroup of G and $\mathrm{Bl}(G \mid P)$ the set of p-blocks with defect group P. Then

$$\mathrm{Irr}_{p'}(G) = \dot{\bigcup_{B \in \mathrm{Bl}(G|P)}} \mathrm{Irr}_0(B).$$

Proof.

Let $\chi \in \mathrm{Irr}_{p'}(G)$ and $B \in \mathrm{Bl}(G)$ be the block with $\chi \in \mathrm{Irr}(B)$.
Then P is the defect group of B.
Hence $\chi \in \mathrm{Irr}_0(B)$.
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Example: 5-blocks of $G = SL_4(\mathbb{F}_2)$

Character Degrees in $\mathrm{Irr}(G)$ (with multiplicities):

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$SL_4(\mathbb{F}_2)$ has seven 5-blocks:

- there are two 5-blocks B_1 and B_2 that have a Sylow 5-subgroup of $SL_4(\mathbb{F}_2)$ as defect group
- every other block contains exactly one character and has 1 as defect group

All characters have height 0.

Note: $\mathrm{Irr}_{5'}(G) = \mathrm{Irr}_0(B_1) \cup \mathrm{Irr}_0(B_2)$.

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Example: 2-blocks of $G = \mathrm{SL}_4(\mathbb{F}_2)$

$\mathrm{SL}_4(\mathbb{F}_2)$ has two 2-blocks:

- B_1 has a Sylow 2-subgroup of $\mathrm{SL}_4(\mathbb{F}_2)$ as defect group
- B_2 has the trivial group as defect group and

degree	64
height	0

Note: $\mathrm{Irr}_{2'}(G) = \mathrm{Irr}_0(B_1)$.

Example: 2-blocks of $G = \mathrm{SL}_4(\mathbb{F}_2)$

$\mathrm{SL}_4(\mathbb{F}_2)$ has two 2-blocks:

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degree	1	7	14	20	21	21	21	28	35	45	45	56	70
height	0	0	1	1	0	0	0	2	0	0	0	2	1

- B_2 has the trivial group as defect group and

degree	64
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Note: $\mathrm{Irr}_{2'}(G) = \mathrm{Irr}_0(B_1)$.

Example: 2-blocks of $G = \mathrm{SL}_4(\mathbb{F}_2)$

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Alperin-McKay conjecture (1975)

Let G be a finite group, B a p-block of G and b the Brauer correspondent of B. Then

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Alperin-McKay conjecture for a 5-block of $G = SL_4(\mathbb{F}_2)$

Recall: The 5-block B_1

character degrees in $\mathrm{Irr}(B_1)$	1	14	21	56	64
height	0	0	0	0	0

$$|\,\mathrm{Irr}_0(B_1)| = 5$$

B_1 has a Sylow 5-subgroup P of $SL_4(\mathbb{F}_2)$ as defect group

Brauer correspondent b_1 of B_1 is a block of $N_G(P)$,

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Recall: The 5-blocks B_2

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height	0	0	0	0

$$|\mathrm{Irr}_0(B_2)| = 4$$

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character degrees in $\mathrm{Irr}(b_2)$	2	2	4	4
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Note $\mathrm{Irr}_{5'}(G) = \mathrm{Irr}_0(B_1) \cup \mathrm{Irr}_0(B_2)$ and $\mathrm{Irr}_{5'}(\mathrm{N}_G(P)) = \mathrm{Irr}_0(b_1) \cup \mathrm{Irr}_0(b_2)$.

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Alperin-McKay conjecture implies the McKay conjecture

Proof.

$$\mathrm{Irr}_{p'}(G) = \dot{\bigcup_{B \in \mathrm{Bl}(G|P)}} \mathrm{Irr}_0(B),$$

where $\mathrm{Bl}(G \mid P)$ is the set of p-blocks of G with defect group P,
Alperin-McKay for B:

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Alperin weight conjecture

Definition

A p-**weight of** G is a pair (Q, ψ), where

- $Q \leq G$ is a p-group, and
- $\psi \in \mathrm{Irr}(\mathrm{N}_G(Q)/Q)$ with $\psi(1)_p = |\,\mathrm{N}_G(Q)/Q|_p$.

Then (Q, ψ) is called a p-**weight of a block** b^G, if ψ as character of $\mathrm{N}_G(Q)$ belongs to $b \in \mathrm{Bl}(\mathrm{N}_G(P))$. ($b^G$ is a p-block of G given by induction of b.)

Alperin weight conjecture (1984)

Let $B \in \mathrm{Bl}(G)$.

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Recall: $\mathrm{SL}_4(\mathbb{F}_2)$ has two 2-blocks, B_1 and B_2

- B_2 has defect 1 and only one Brauer character.
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Let B be a p-block of G and D its defect group.
Every character of B has height zero if and only if D is abelian.

Example: G is a p-group

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$\mathrm{SL}_4(\mathbb{F}_2)$ has two 2-blocks:

- B_1 has a Sylow 2-subgroup P as defect group, where

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degree	1	7	14	20	21	21	21	28	35	45	45	56	70
height	0	0	1	1	0	0	0	2	0	0	0	2	1

The defect group is non-abelian and $\mathrm{Irr}_0(B_1) \neq \mathrm{Irr}(B_1)$.

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Reductions and Results

Theorem

All mentioned conjectures are true for p-solvable groups

A group is p-**solvable** if all its simple composition factors are cyclic or a p'-group.

Furthermore results are known for specific blocks and families of groups.

Knowledge on nonabelian quasisimple groups seems missing.

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One direction of Brauer's height zero conjecture

Definition

A group is **quasisimple** if $G = [G, G]$ and $G / \mathrm{Z}(G)$ is simple.

Theorem (HZC1) (Berger-Knörr, Kessar-Malle)

Let G be a finite group, $B \in \mathrm{Bl}(G)$ and D a defect group. Then

$$D \text{ is abelian} \Rightarrow \mathrm{Irr}_0(B) = \mathrm{Irr}(B).$$

History of the proof:

Berger-Knörr (1988): "HZC1" is true, if it holds for all quasisimple groups.

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Reductions - McKay and Alperin weight conjecture

Theorem (Isaacs-Malle-Navarro, 2007)

The McKay conjecture is true if all quasisimple groups are McKay-good.

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The non-blockwise version of Alperin's weight conjecture is true if all quasisimple group are AWC(Alperin-weight)-good.

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Theorem (S., 2011)

The Alperin-McKay conjecture is true if all quasisimple groups are AM-good.

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When is a quasisimple group good?

Philosophy:

- There is a correspondence between the considered sets of characters that is equivariant with respect to automorphisms of the group stabilizing those sets.
- Characters χ and χ' associated to each other via this correspondence have the same "Clifford-theory", i.e. after inducing χ and χ' to some groups they decompose in the same way.

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Let p be a prime, $\widehat{S}$ a quasisimple group and B a block of $\widehat{S}$.

Definition

The block B is **AM-good**, if the defect group of B is central.
The block B is **AM-good**, if for the non-central defect group D of B

1. there exists a group M with
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AM-good and BAW-good groups

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A quasisimple group $\widehat{S}$ is AM-good, if all blocks B of $\widehat{S}$ are AM-good, i.e. the above strengthened equivariant version of the Alperin-McKay conjecture holds for B.

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$SL_4(\mathbb{F}_2)$ is AM-good for $p = 2$

$$\mathrm{Aut}(SL_4(\mathbb{F}_2)) = SL_4(\mathbb{F}_2) \rtimes \langle \Gamma \rangle$$

Here Γ is given by $x \mapsto J(x^{\perp})^{-1}J$, where J corresponds to the longest element of $\mathcal{S}_n$.

Take

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The last condition is trivial if χ is not Γ-invariant.

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Note that χ extends to A and $\Omega(\chi)$ extends to $P \rtimes \langle \Gamma \rangle$. Hence the last condition is satisfied here as well.

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Projective representation

We call $\mathcal{P} : G \to \mathrm{GL}_n(\mathbb{C})$ **a projective representations**, if the induced map $G \to \mathrm{PGL}_n(\mathbb{C})$ is a group morphism.

For a projective representation $\mathcal{P}$ there exists a map $\alpha : G \times G \to \mathbb{C}^*$ with

$$\mathcal{P}(g)\mathcal{P}(g') = \alpha(g, g')\mathcal{P}(gg') \text{ for every } g, g' \in G.$$

This map $\alpha : G \times G \to \mathbb{C}^*$ is the **factor set of** $\mathcal{P}$.

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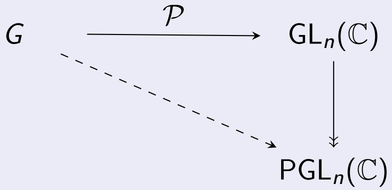

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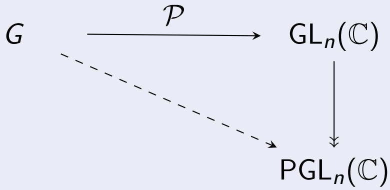

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Extending to projective representation of G_θ:
We find a projective representation $\mathcal{P}$ of some G_θ, such that $\mathcal{P}|_N$ is a representation affording θ and the factor set of $\mathcal{P}$ is constant on $N \times N$-cosets.

Tensoring with projective representation of G_θ/N:
We can tensor $\mathcal{P}$ with a (irreducible) projective representation of G_θ/N whose factor set is inverse to the one of $\mathcal{P}$. Then $\mathcal{P} \otimes \mathcal{Q}$ is a representation of G_θ.

Induction of character to G:
Let ψ be a character afforded by $\mathcal{P} \otimes \mathcal{Q}$ then $\mathrm{Ind}_{G_\theta}^G(\psi)$ is irreducible.

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Implications of AM-Goodness

Theorem (Navarro-S.)

Let $B \in \mathrm{Bl}(G)$ with defect group D and $b \in \mathrm{Bl}(\mathrm{N}_G(D))$ the Brauer correspondent of b. Suppose that G is a normal subgroup of A. Assume that all non-abelian simple groups involved in G and their extensions are AM-good.

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Let $\widehat{S}$ be a quasisimple group such that $\widehat{S}/\mathrm{Z}(\widehat{S})$ is a simple group of Lie type. Let p be a prime different from the characteristic of the field underlying $\widehat{S}$ and P a Sylow p-subgroup of $\widehat{S}$. Then there exists a group $N \geq \mathrm{N}_{\widehat{S}}(P)$ and a bijection

$$\Omega : \mathrm{Irr}_{p'}(\widehat{S}) \longrightarrow \mathrm{Irr}_{p'}(N)$$

Problem: Ω is given by parametrizing the sets with the same labels (but labelling depends on choices!)

Theorem (Cabanes-S., 2012)

1. *Quasisimple groups of type* ${}^3\mathrm{D}_4$, E_8, F_4, ${}^2\mathrm{F}_4$ *and* G_2 *are McKay-good.*
2. *Let* $\widehat{S}$ *be a quasisimple group of type* ${}^3\mathrm{D}_4$ E_8, F_4 *or* G_2. *For* $\ell \geq 5$ *the blocks of* $\widehat{S}$ *whose defect groups are Sylow p-subgroups are BAW-good.*
3. *Let* $\widehat{S}$ *be a quasisimple group of type of* ${}^3\mathrm{D}_4$ E_8, F_4 *or* G_2. *For* $\ell \geq 7$ *the blocks of* $\widehat{S}$ *whose defect groups are Sylow p-subgroups are AM-good.*

Theorem (Cabanes-S., 2013)

The groups $\mathrm{SL}_n(q)$ *and* $\mathrm{SU}_n(q)$ *and their quotients are McKay-good.*

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Challenges with $\widehat{S} := \mathrm{SL}_n(\mathbb{F}_q)$ (and $p \nmid q$)

Automorphisms of $\mathrm{SL}_n(\mathbb{F}_q)$:

- $\mathrm{GL}_n(\mathbb{F}_q)$ acts on $\mathrm{SL}_n(\mathbb{F}_q)$ by conjugation
- graph automorphism Γ with $g \mapsto (g^{\perp})^{-1}$
- Galois automorphisms of $\mathbb{F}_q$ induce automorphisms on $\mathrm{SL}_n(\mathbb{F}_q)$

There is a torus $\mathbf{T}$ of $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$ such that $\mathrm{N}_{\mathrm{SL}_n(\mathbb{F}_q)}(\mathbf{T})$ plays the role of $\mathrm{N}_{\widehat{S}}(P)$ for a Sylow p-subgroup P.
Indirect proof of the McKay-goodness uses:

- Special shape of stabilizers of characters (not every subgroup of $\mathrm{Aut}(\mathrm{SL}_n(\mathbb{F}_q))$ occurs as the stabilizer of a character of $\mathrm{SL}_n(\mathbb{F}_q)$ (proven by use of Kawanaka's generalized Gelfand-Graev characters)
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Corollary

Let $\widehat{S}$ be a quasisimple quotient of $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$. The blocks of $\widehat{S}$ whose defect groups are Sylow p-subgroups are AM-good.

Hope:

A key ingredient on GGGR's is missing in other types, but some of the results can be transferred to other types.

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Thank you!