The Lusztig Conjecture

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on
on irreduzible characters
of algebraic groups

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- Dimensions of its weight spaces for a maximal torus?

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$$L \mapsto (\dim L) - 1$$

$$k[X, Y]^{(n)} \leftarrow\!\shortmid n$$

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For affine algebraic groups $G \supset B$ the restriction admits a right adjoint, induction

$$G\text{-Mod} \underset{\text{ind}}{\overset{\text{res}}{\rightleftarrows}} B\text{-Mod}$$

$$\begin{aligned}\operatorname{ind}_B^G V &= \{f : G \to V \mid f \text{ algebraic } B\text{-equivariant}\} \\ &= \{ \text{ algebraic sections in } G \times_B V \twoheadrightarrow G/B\}\end{aligned}$$

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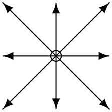
Example $G = \mathrm{Sp}(4; k)$



$\mathfrak{X}$ weight lattice

$\mathfrak{X}^+$ dominant weights

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Steinberg tensor product theorem:

$G \supset B$ and $\mathfrak{X} \supset \mathfrak{X}^+$ general again.
For $\lambda \in \mathfrak{X}^+$ consider the p-adic expansion

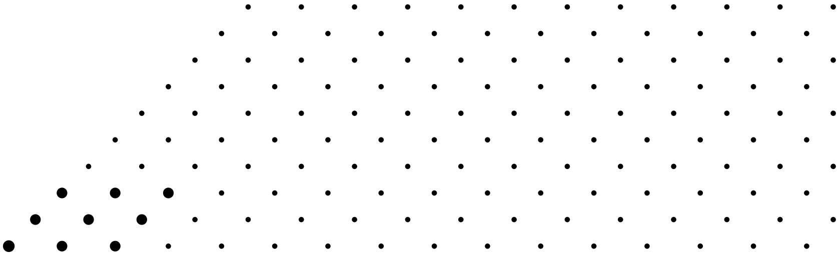
$$\lambda = p^d \lambda_d + \ldots + p^2 \lambda_2 + p\lambda_1 + \lambda_0$$

with digits λ_i in the fundamental box, given by
Box$:= \{\mu \in \mathfrak{X}^+ \mid \langle \mu, \alpha^\vee \rangle < p$ for all simple roots $\alpha\}$

Then we have

$$L(\lambda) \cong L(\lambda_d)^{[d]} \otimes \ldots \otimes L(\lambda_2)^{[2]} \otimes L(\lambda_1)^{[1]} \otimes L(\lambda_0)$$

Here $L^{[i]}$ is the twist of L by the i-th power of the Frobenius automorphism of $\mathrm{GL}(L)$.



The 9 elements of box in case $p = 3$ and $G = \mathrm{Sp}(4; k)$

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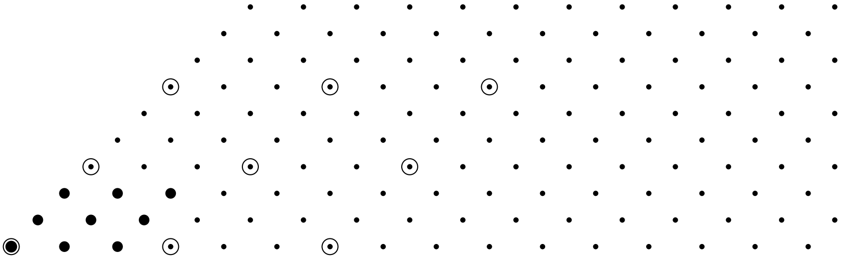

The 9 elements of box in case $p = 3$ and $G = \mathrm{Sp}(4; k)$ along with the $p\lambda_1$ for $\lambda_1 \in$ Box

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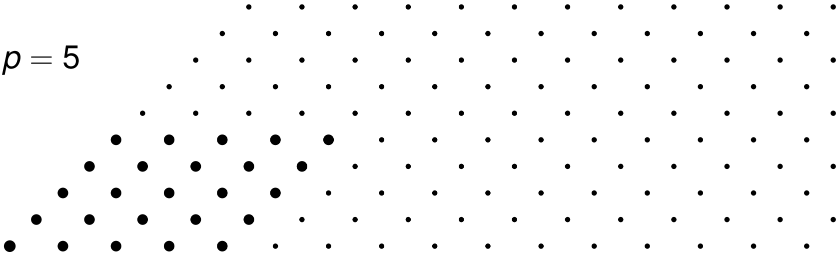
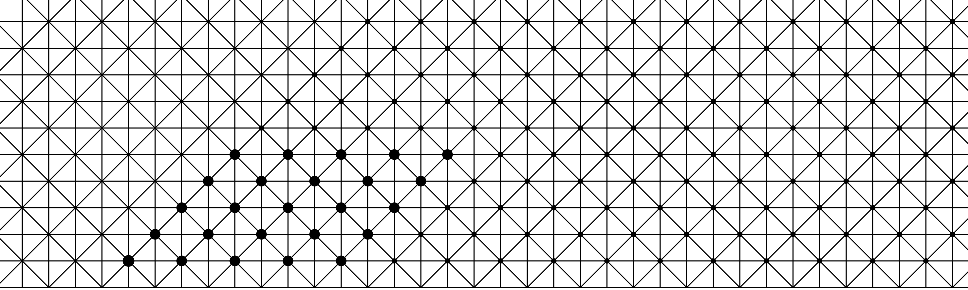


Consider affine Weyl group $\mathcal{W} = W \ltimes \langle R \rangle$



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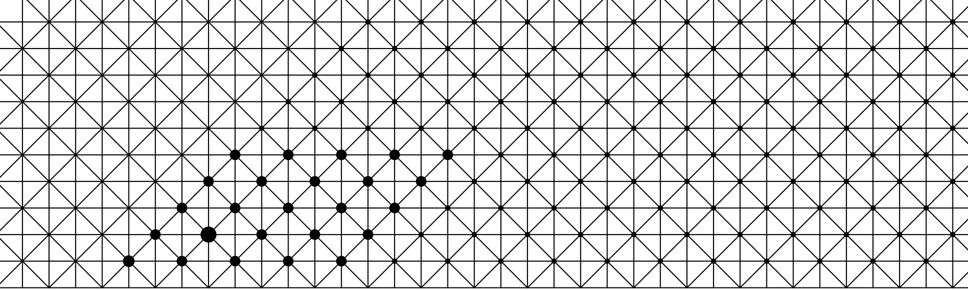

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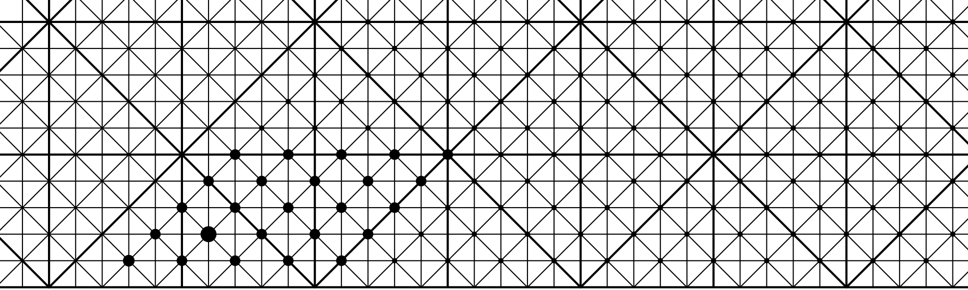

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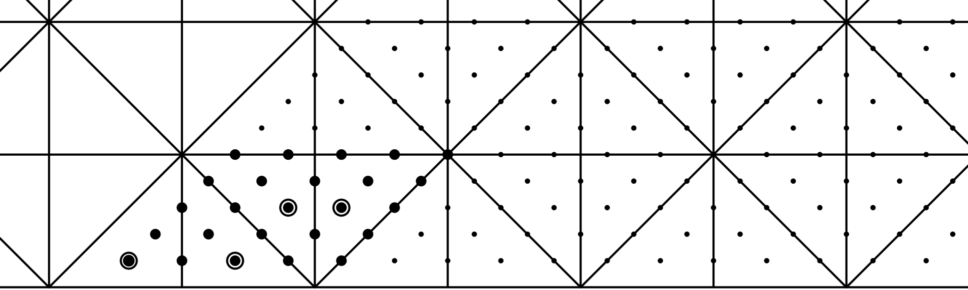

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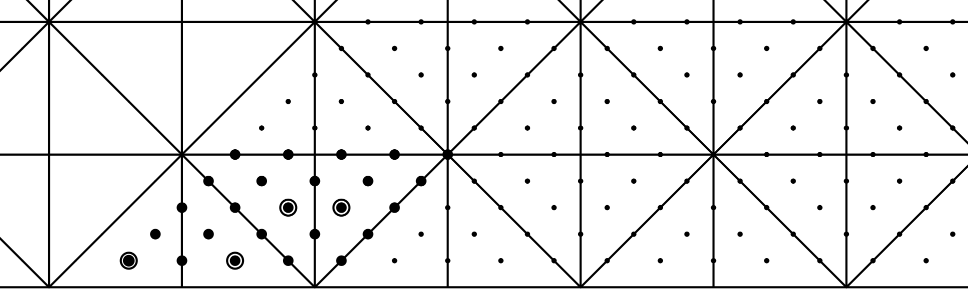
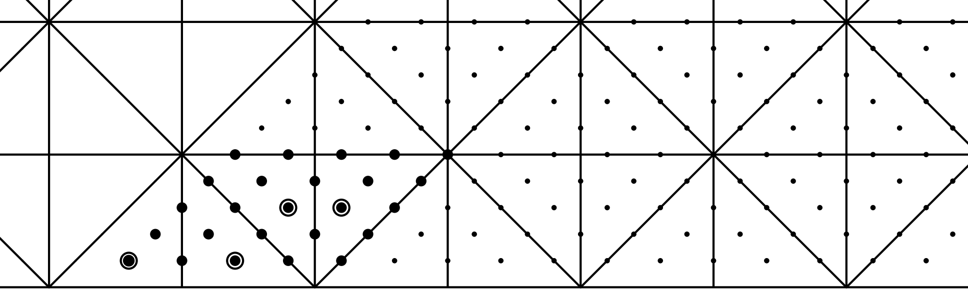


Lusztig conjecture

For $x \in \mathcal{W}$ with $x \cdot_p 0 \in$ Box and p so big,
that $z \cdot_p 0 = 0 \Rightarrow z = 1$ we should have:

$$[L(x \cdot_p 0)] = \sum_y (-1)^{l(x)+l(y)} P_{w_\circ y, w_\circ x}(1)\ [\nabla(y \cdot_p 0)]$$

Translation principle: These $[L(x \cdot_p 0)]$ give all $[L(\lambda)]$ for $\lambda \in$ Box



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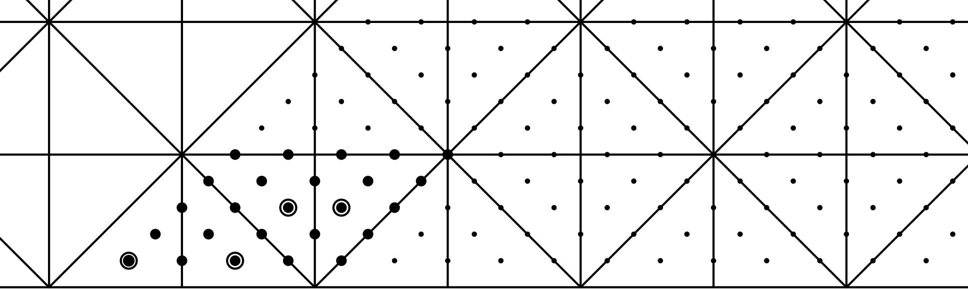

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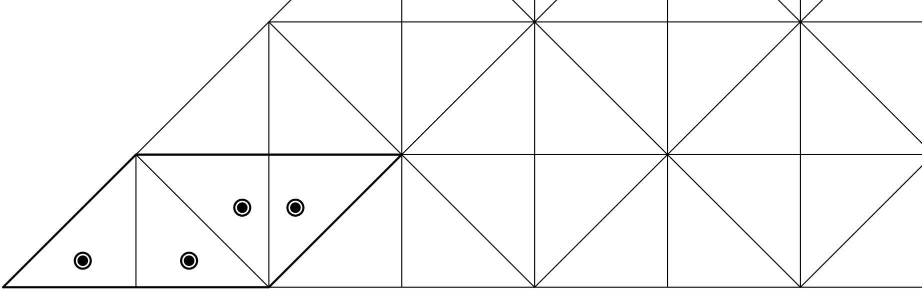

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$L_A := L(x \cdot_p 0)$, $\nabla_B := \nabla(y \cdot_p 0)$ and $m_{B,A} := P_{w_\circ y, w_\circ x}$

Lusztig conjecture: For A in the fundamental box should have

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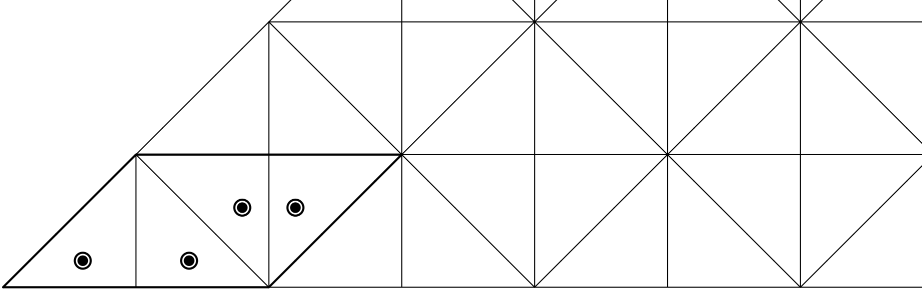

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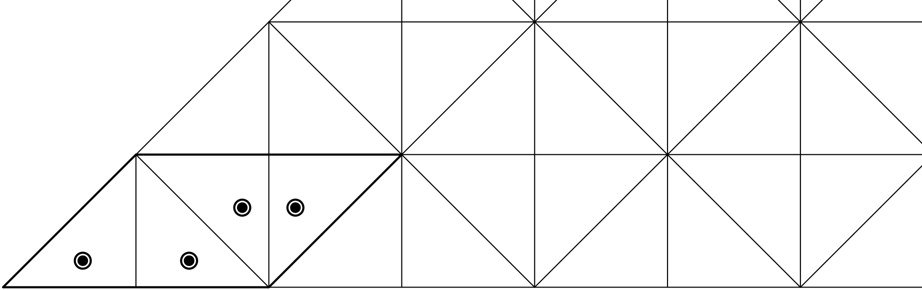

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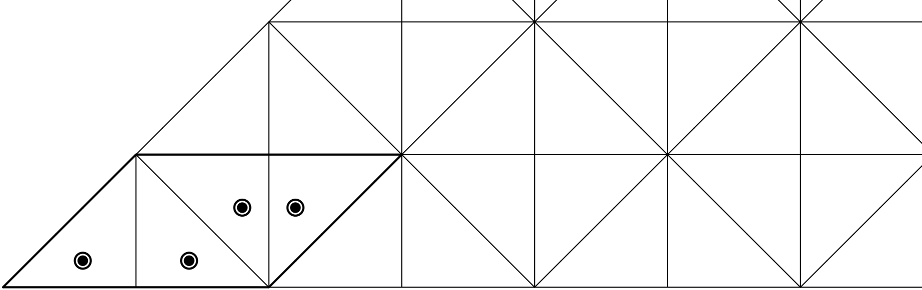

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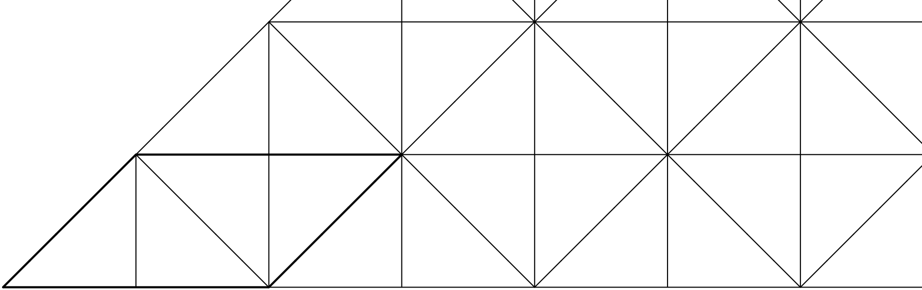

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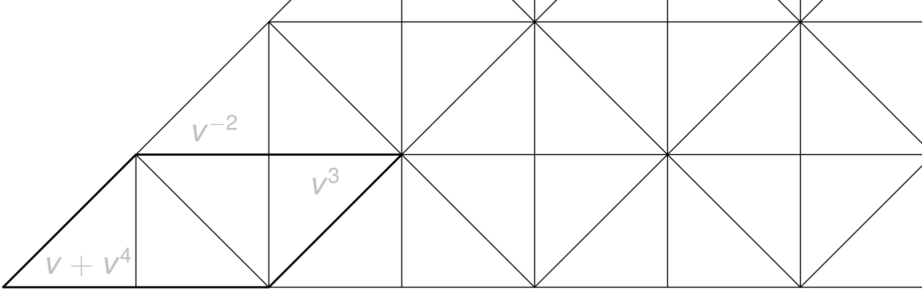

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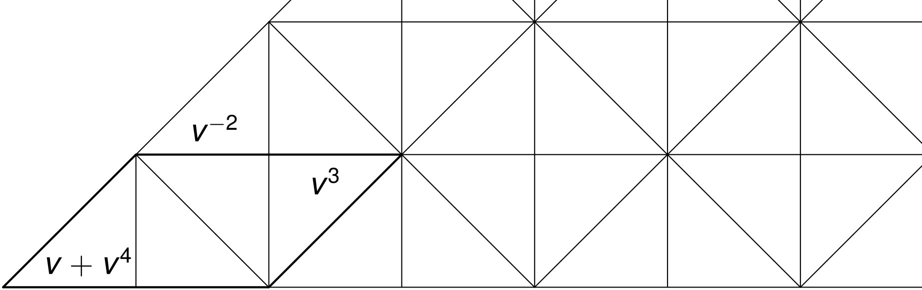

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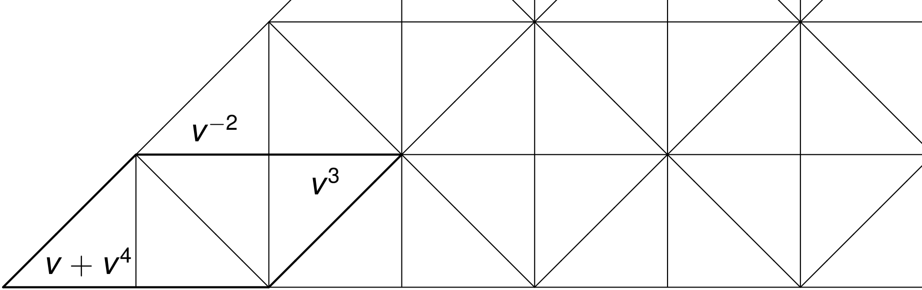

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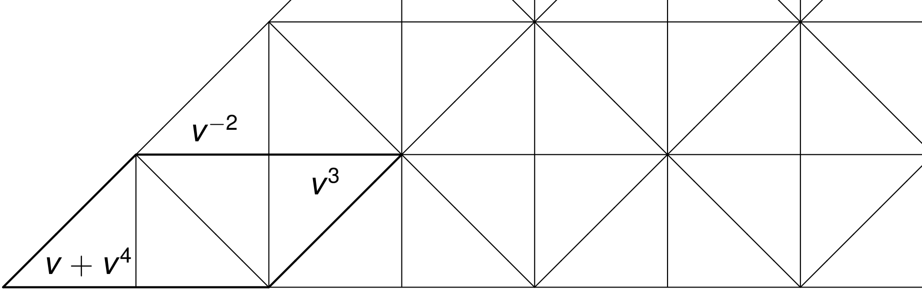

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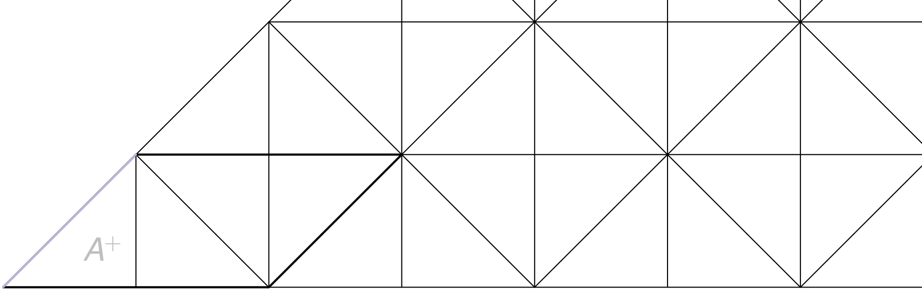

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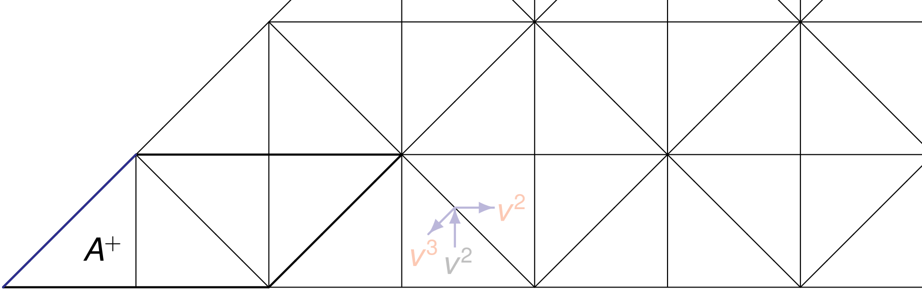

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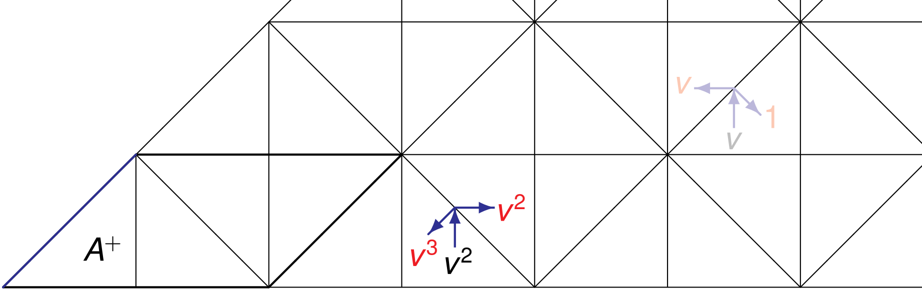

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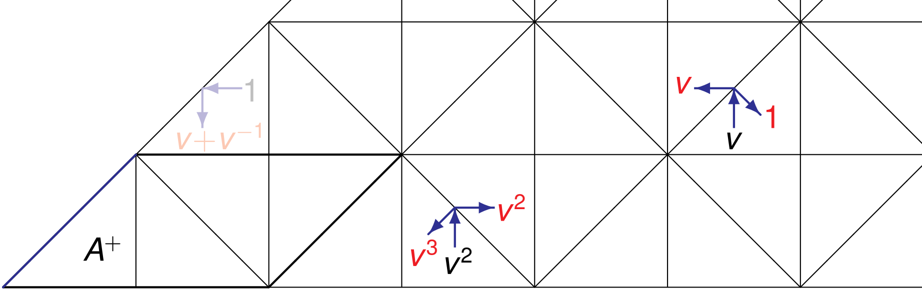

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$$[s] : A \mapsto As + vA \quad \text{in case } As > A \text{ and } As \in \mathcal{A}^+;$$
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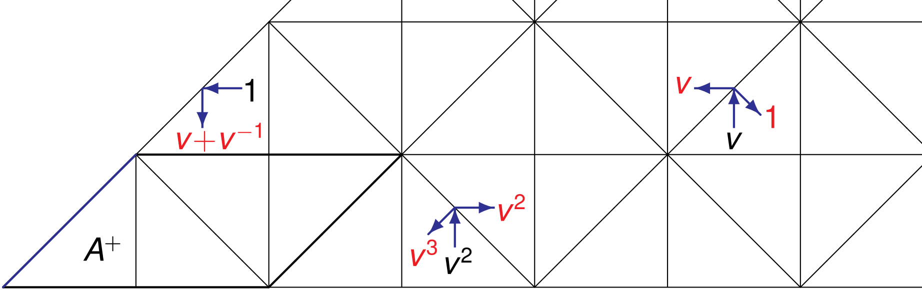

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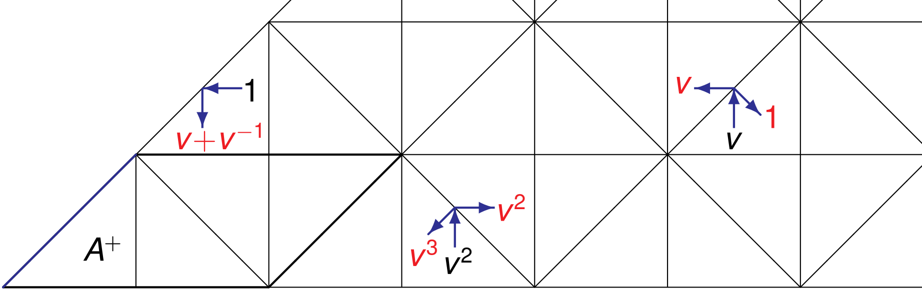

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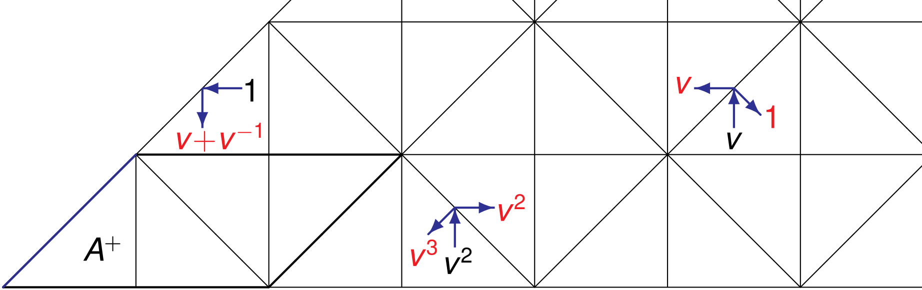

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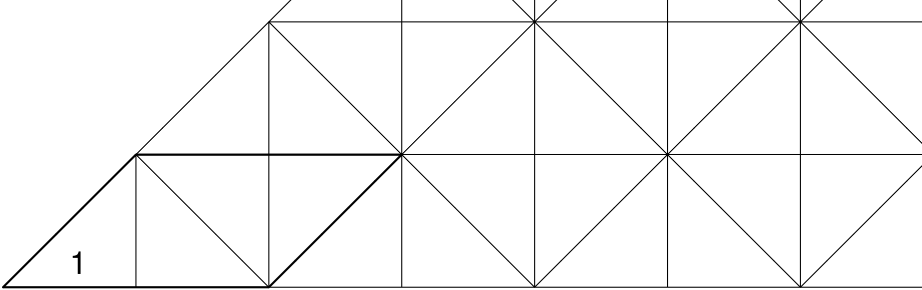

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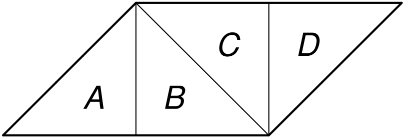
$[L_A] = [\nabla_A]$ the trivial representation



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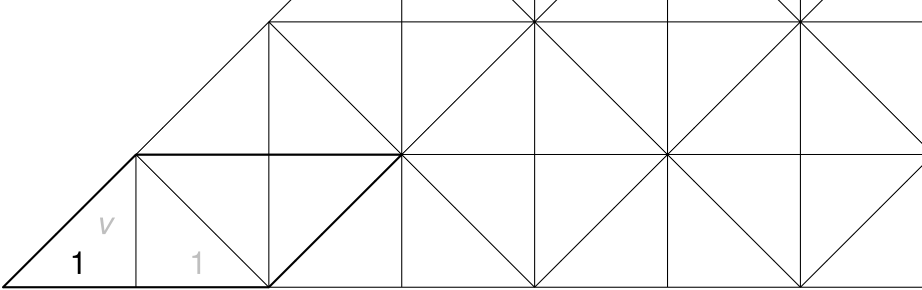

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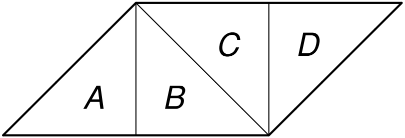

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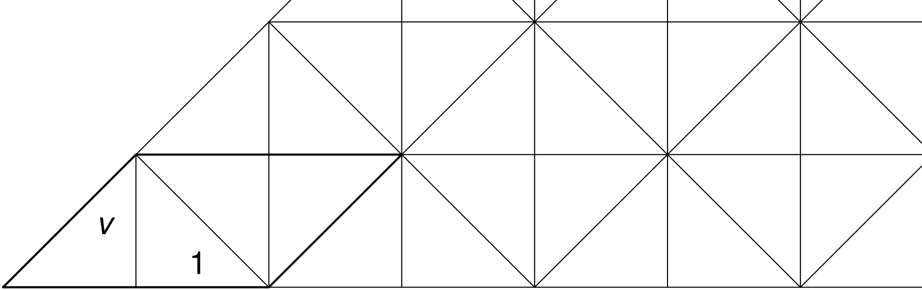

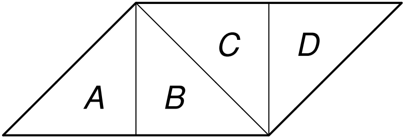

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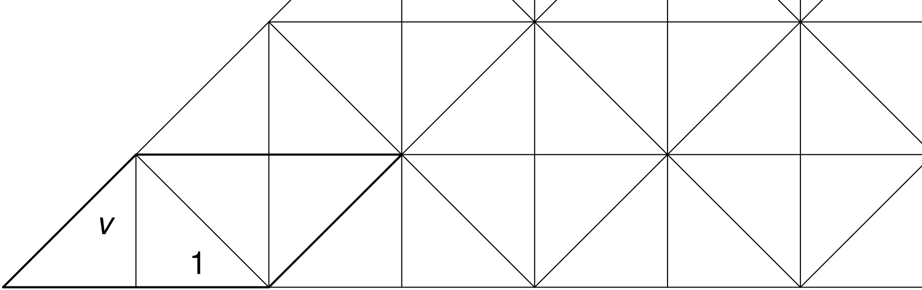
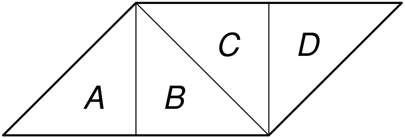


$\underline{M}_B = B + vA$

$[L_B] = [\nabla_B] - [\nabla_A]$



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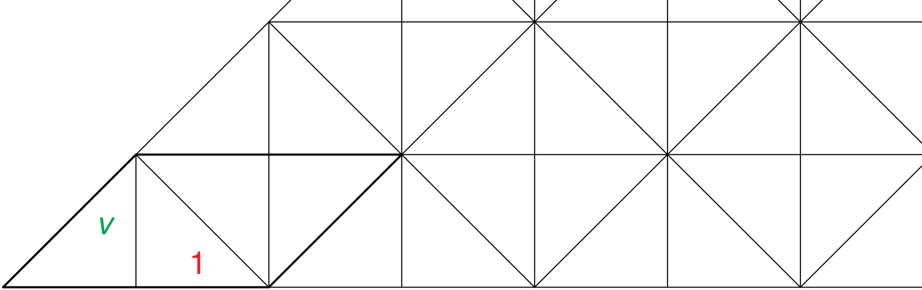

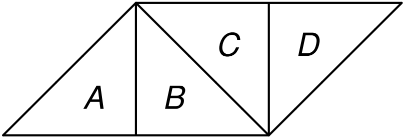

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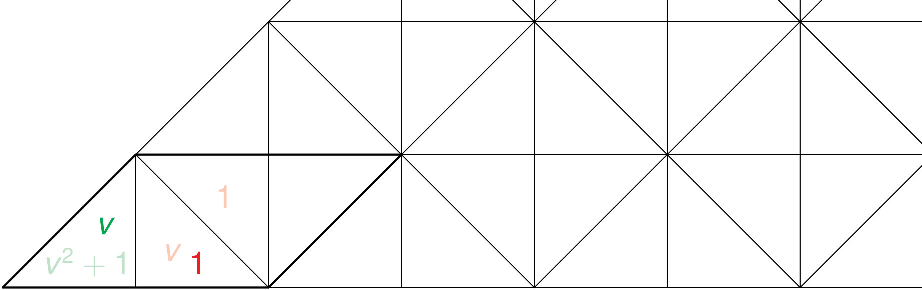

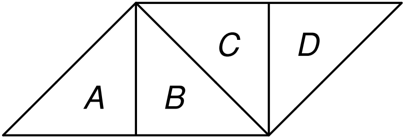

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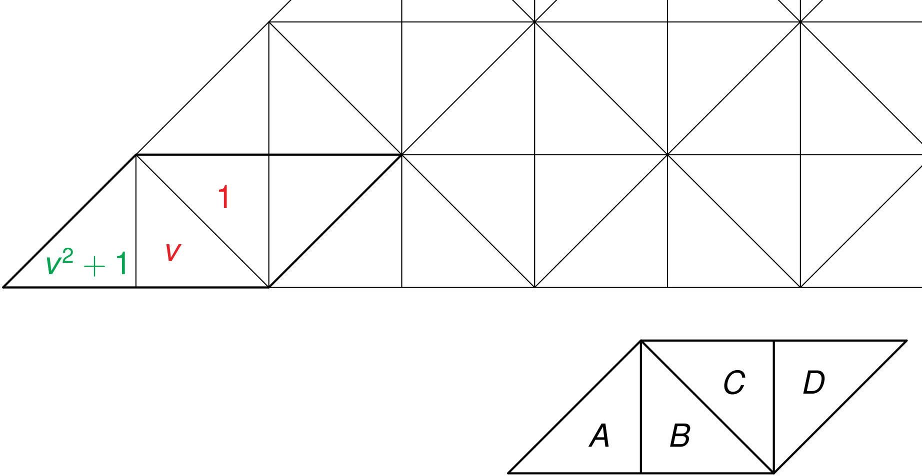

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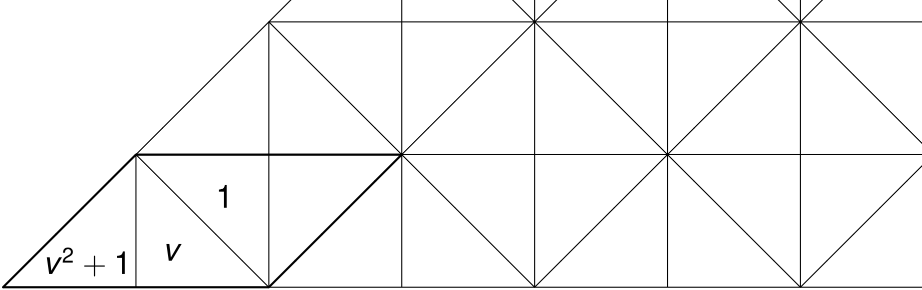

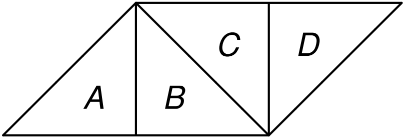

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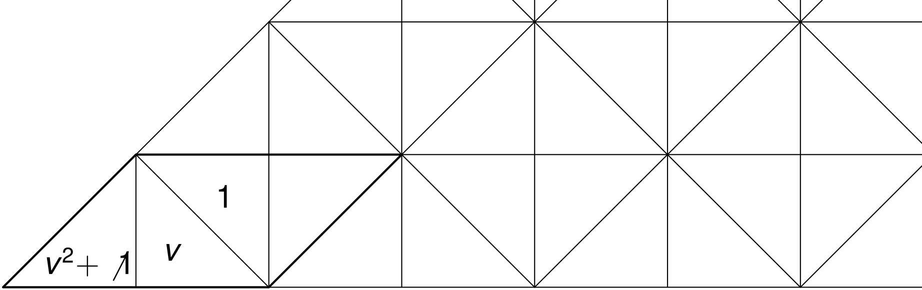

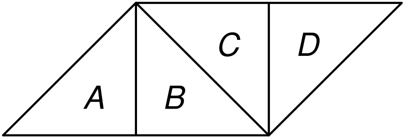

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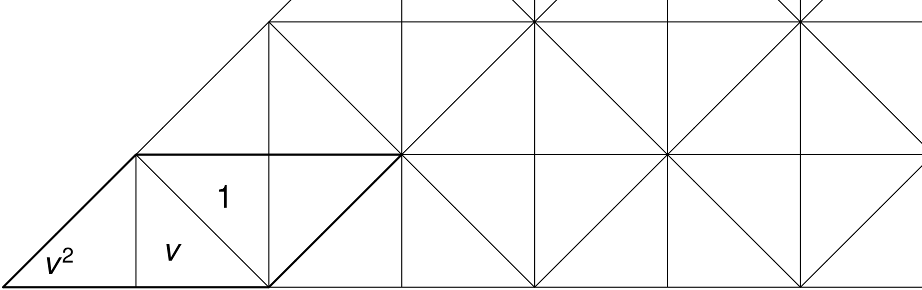
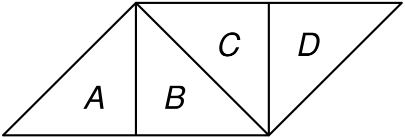


$\underline{M}_C = C + vB + v^2 A$

$[L_C] = [\nabla_C] - [\nabla_B] + [\nabla_A]$



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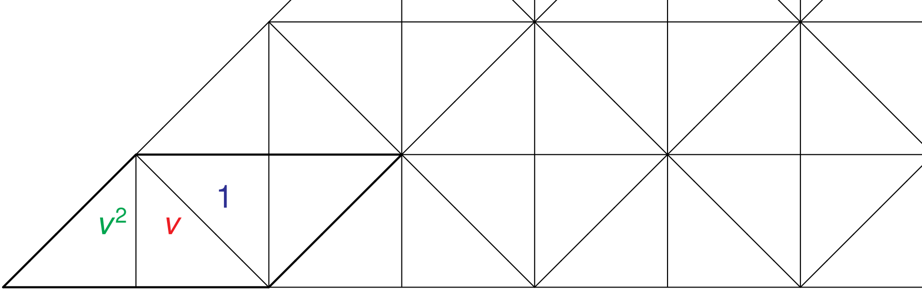

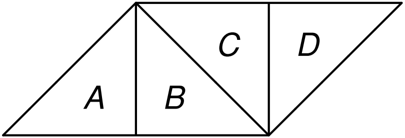

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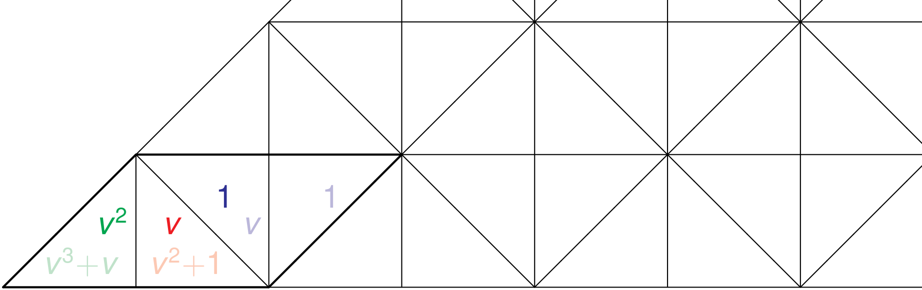

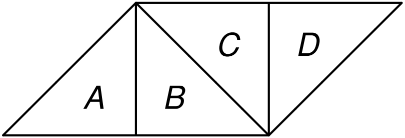

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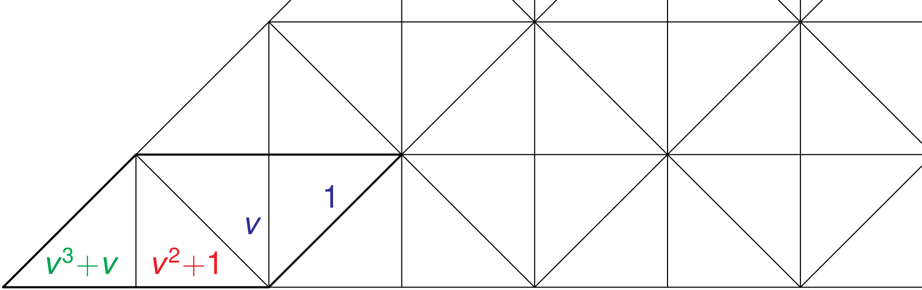

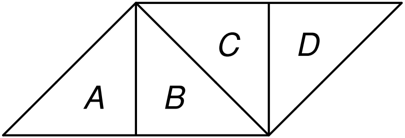

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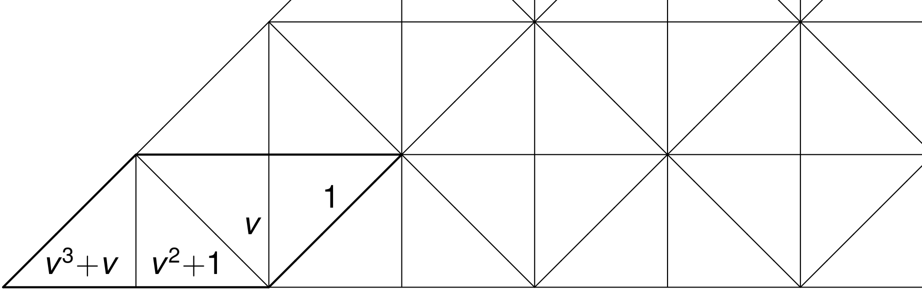

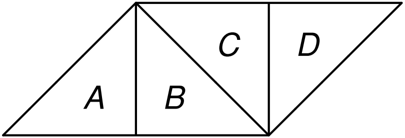

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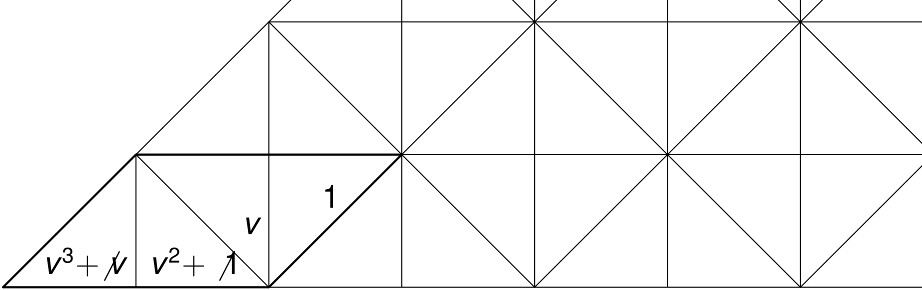

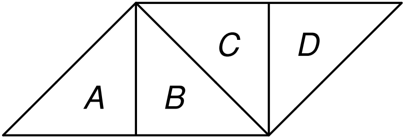

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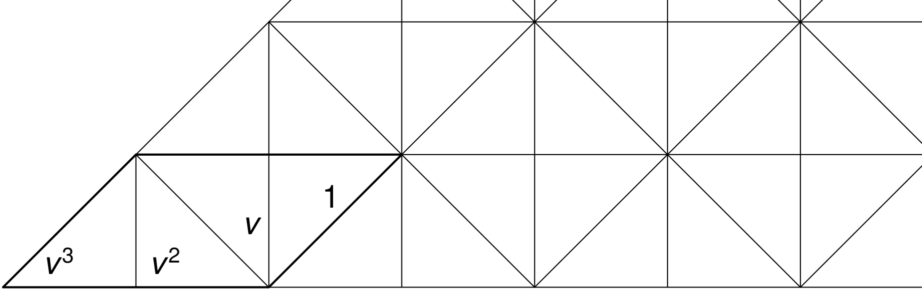
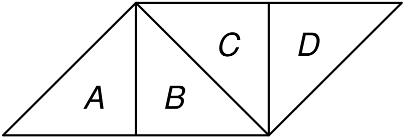


$$\underline{M}_D = D + vC + v^2B + v^3A$$

$$[L_D] = [\nabla_D] - [\nabla_C] + [\nabla_B] - [\nabla_A]$$



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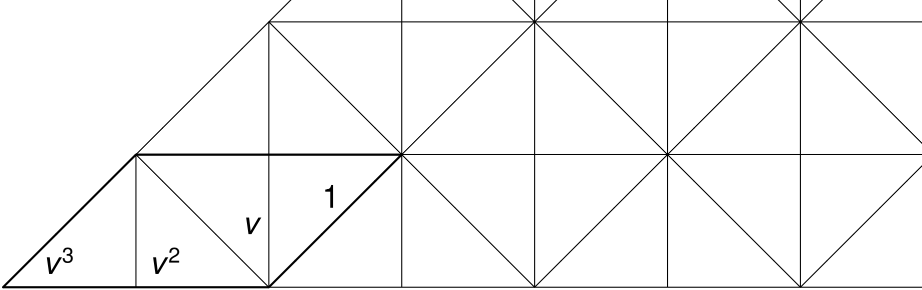

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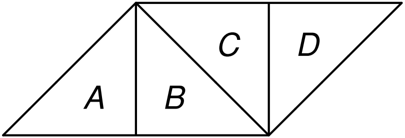

THANK YOU!

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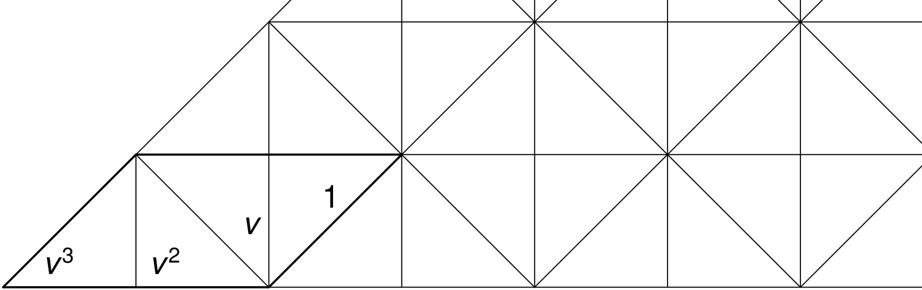

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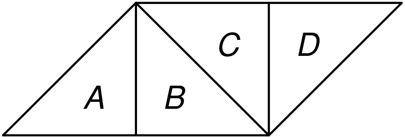

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