On Loewy lengths of blocks (joint work with S. Koshitani and B. Külshammer)

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- $J(B)$ – Jacobson radical of B (as an algebra)
- Let $LL(B) := \min\{n \geq 0 : J(B)^n = 0\}$ be the Loewy length of B
- Let D be a defect group of B. This is p-subgroup of G, unique up to conjugation.

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   (a) $p = \delta = 2$ *and B is Morita equivalent to* $F[C_2 \times C_2]$ *or to* FA_4.
   (b) $p > 2$, $\delta = 1$, *the inertial index of B is* $e(B) \in \{p-1, (p-1)/2\}$, *and the Brauer tree of B is a straight line with exceptional vertex at the end (if it exists).*

Theorem (Koshitani-Külshammer-S.)

If B has defect δ and $LL(B) > 1$, then

$$\delta \leq \binom{LL(B)}{2}(2\lfloor \log_p(LL(B) - 1)\rfloor + 1).$$

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- Combine these equations. □

Remarks

Brauer's Problem 21

Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} f(n) = \infty$ and $f(\delta) \leq \dim_F Z(B)$.

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Proposition

Let B be a block with cyclic defect group D and inertial index $e(B)$. Then

$$LL(B) \geq \frac{|D| - 1}{e(B)} + 1.$$

Blocks with $LL(B) = 4$

Proposition

Let B be a p-block with defect δ, defect group D and $LL(B) = 4$. Then

$$\delta \le \begin{cases} 18 & \text{if } p \le 3, \\ 5 & \text{if } p = 5, \\ 6 & \text{if } p \ge 7. \end{cases}$$

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In case $p = 5$ (resp. $p = 7$) there are at most 10 (resp. 12) isomorphism types for D. These can be given by generators and relations. All these groups have exponent p and rank at most 3.

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Theorem

Let $G = S_n$ and $LL(B) = 4$. Then $n = 4$ and B is the principal 2-block.

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We do not know if $LL(B_0(M)) = 4$ for $p = 11$ (probably not).

Examples

- Let $p \equiv 1 \pmod 3$, $n := (p-1)/3$ and $G := \mathrm{PSL}(n, q)$ where q has order n modulo p, but not modulo p^2 (q always exists). Then $LL(B_0(G)) = 4$.

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  - $G = 3.O'N$ for $p = 5$,
  - $G = Ru$ and $G = 2.Ru$ for $p = 7$.
- We do not have any examples for $p = 3$.