Ideals in $U(\mathfrak{g})$ for locally simple Lie algebras $\mathfrak{g}$

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Base field $\mathbb{C}$.
Locally simple Lie algebras:

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_n \subset \mathfrak{g}_{n+1} \cdots$$

$\mathfrak{g}_n$ simple, $\lim_{n\to\infty}(\dim \mathfrak{g}_n) = \infty$, $\mathfrak{g} = \varinjlim_{n\in\mathbb{Z}_{\geq 1}} \mathfrak{g}_n$.
Three classes:

I. $\mathfrak{g}$ — *finitary* (classical):

$$\mathfrak{sl}(\infty) = \varinjlim_{n\in\mathbb{Z}_{\geq 2}} \mathfrak{sl}(n),$$

$$\mathfrak{o}(\infty) = \varinjlim_{n\in\mathbb{Z}_{\geq 3}} \mathfrak{o}(n),$$

$$\mathfrak{sp}(\infty) = \varinjlim_{n\in\mathbb{Z}_{\geq 2}} \mathfrak{sp}(2n).$$

II. $\mathfrak{g}$ — *diagonal*: example $\mathfrak{sl}(2^\infty) = \varinjlim_{n\in\mathbb{Z}_{\geq 1}} \mathfrak{sl}(2^n)$

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

Classification by A.Baranov and A.Zhilinskii
(A. A. Baranov, A. G. Zhilinskii, Diagonal direct limits of semisimple Lie algebras, Comm. Algebra **27** (1999), 2749-2766)

III. $\mathfrak{g}$ — *general non-diagonal*: example $\mathrm{adj}_n(\infty)$

$$\mathfrak{sl}(n) \subset \mathfrak{sl}(n^2-1) \subset \ldots$$

No classification

$\mathfrak{g}$ admits no non-trivial finite-dimensional representations and $Z_{U(\mathfrak{g})} = \mathbb{C}$.

$I \subset U(\mathfrak{g})$, degree filtration yields $\mathrm{gr}\, I \subset S^{\cdot}(\mathfrak{g})$, $\mathrm{gr}\, I$ is G-stable (Poisson)

$U(\mathfrak{g})$ admits a non-zero ideal of locally infinite codimension

$$\Downarrow$$

$S^{\cdot}(\mathfrak{g})$ admits a non-zero Poisson ideal of locally infinite codimension

associated variety $\mathrm{Var}(I) =$ zeroes of $\mathrm{gr} I$

$$\mathrm{Var}(I) \subset \mathfrak{g}^* = \varprojlim_{n \in \mathbb{Z}_{\geq 1}} \mathfrak{g}_n^*$$

I has locally finite codimension $\Leftrightarrow \mathrm{Var}(I) = 0$

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Theorem (first main result)

If $S^{\cdot}(\mathfrak{g})$ admits a non-zero Poisson ideal of locally infinite codimension, then $\mathfrak{g} \simeq \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$.

Theorem (A. G. Zhilinskii, Coherent finite-type systems of inductive families of non-diagonal inclusions, (Russian) Dokl. Acad. Nauk Belarusi 36:1 (1992), 9-13, 92)

If $U(\mathfrak{g})$ admits a non-zero ideal of locally infinite codimension, then $\mathfrak{g}$ is diagonal.

Corollary (Baranov's conjecture)

If $\mathfrak{g}$ is not diagonal, then the augmentation ideal is the only proper ideal in $U(\mathfrak{g})$.

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Idea of proof of Theorem 1. Let $I_n = I \cap U(\mathfrak{g}_n)$. We have

$$\begin{array}{ccc} \mathrm{Var}(I_{n+m}) & \subset & \mathfrak{g}^*_{n+m} \\ \downarrow & & \downarrow p_{n,m} \\ \mathrm{Var}(I_n) & \subset & \mathfrak{g}^*_n \end{array}$$

Key Proposition: $O \subset \mathfrak{g}^*_{n+m}, O - G_{n+m} - orbit, p_{n,m}(O)$ is not dense in $\mathrm{g}^*_n \quad \Rightarrow$

$$\dim(\mathfrak{g}_n V_{n+m}) < 2(\dim G_n - rkG_n)(rkG_n + 1)$$

or

$$2 \dim G_n + 2 \geq \dim V_{m+n}$$

Identify: $\mathfrak{g}_n^* = \mathfrak{g}_n$. Set

$$\mathfrak{g}_n^{\leq r} = \{X \in \mathfrak{g}_n | \exists \lambda \in \mathbb{C} : rk(X - \lambda \mathrm{Id}_{V_n}) \leq r\}.$$

We have a well-defined projective system

$$\ldots \to \mathfrak{g}_n^{\leq r} \to \mathfrak{g}_{n-1}^{\leq r} \to \ldots$$

for $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$. In fact, the above maps are surjective. We obtain Poisson ideals $J^{\leq r}$ of $S^{\cdot}(\mathfrak{g})$ whose zero-sets are respectively $\varprojlim_{n \in \mathbb{Z}_{\geq 1}} \mathfrak{g}_n^{\leq r} \subset \mathfrak{g}^*$.

Theorem (second main result)

Let $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ and let $J \subset S^{\cdot}(\mathfrak{g})$ be a non-zero radical Poisson ideal. Then $J = J^{\leq r}$ for some r.

In the remainder $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$.

Definition

$I \subset U(\mathfrak{g})$ is *integrable* if for any n, $I_n = I \cap U(\mathfrak{g}_n)$ is the intersection of ideals of finite codimension in $U(\mathfrak{g}_n)$.

Definition (Zhilinskii)

A *coherent local system* (cls) of simple finite-dimensional $\mathfrak{g}_\mathfrak{n}$-modules is a set $\{M_n^k\}$ such that

$$\forall n, k \;\; M^k_{n\downarrow\mathfrak{g}_{n-1}} = \bigoplus_{k'} m^{k'} M^{k'}_{n-1}.$$

A cls is of *finite type* if for any fixed n it contains finitely many $\mathfrak{g}_n$-modules M_n^k.
A cls is *irreducible* if it is not the union of two cls.

The following are *basic* cls:

for $\mathfrak{g} = \mathfrak{sl}(\infty) : \mathcal{E} \leftarrow \Lambda^{\cdot} V,\ \mathcal{L}_p \leftarrow \Lambda^p V,\ \mathcal{L}_p^{\infty} \leftarrow S^{\cdot}(V \otimes \mathbb{C}^p)$,
$\mathcal{R}_q \leftarrow \Lambda^q(V_*),\ \mathcal{R}_q^{\infty} \leftarrow S^{\cdot}((V_*) \otimes \mathbb{C}^q),\ \mathcal{E}^{\infty}$ (all modules);

for $\mathfrak{g} = \mathfrak{o}(\infty) : \mathcal{E} \leftarrow \Lambda^{\cdot} V,\ \mathcal{L}_p \leftarrow \Lambda^p V,\ \mathcal{L}_p^{\infty} \leftarrow S^{\cdot}(V \otimes \mathbb{C}^p)$,
$\mathcal{R}$ (spinor modules), $\mathcal{E}^{\infty}$ (all modules);

for $\mathfrak{g} = \mathfrak{sp}(\infty) : \mathcal{E} \leftarrow \Lambda^{\cdot} V,\ \mathcal{L}_p \leftarrow \Lambda^p V,\ \mathcal{L}_p^{\infty} \leftarrow S^{\cdot}(V \otimes \mathbb{C}^p)$,
$\mathcal{E}^{\infty}$ (all modules);

where $p, q \in \mathbb{Z}_{\geq 1}$

Product and tensor product of cls are well-defined. Moreover, any cls has the form

$$(\mathcal{L}_1^\infty)^{\otimes \upsilon} \otimes (\mathcal{R}_1^\infty)^{\otimes w} \otimes ((\mathcal{L}_1^{x_{\upsilon+1}} \mathcal{L}_2^{x_{\upsilon+2}} \ldots \mathcal{L}_{i-\upsilon}^{x_i}) \mathcal{E}^m (\mathcal{R}_1^{z_{w+1}} \mathcal{R}_2^{z_{w+2}} \ldots \mathcal{R}_{j-w}^{z_j}))$$

for $\mathfrak{g} = \mathfrak{sl}(\infty)$, and

$$(\mathcal{L}_1^\infty)^{\otimes \upsilon} \otimes ((\mathcal{L}_1^{x_{\upsilon+1}} \mathcal{L}_2^{x_{\upsilon+2}} \ldots \mathcal{L}_{i-\upsilon}^{x_i}) \mathcal{E}^m \mathcal{R})$$

for $\mathfrak{g} = \mathfrak{o}(\infty)$,

$$(\mathcal{L}_1^\infty)^{\otimes \upsilon} \otimes ((\mathcal{L}_1^{x_{\upsilon+1}} \mathcal{L}_2^{x_{\upsilon+2}} \ldots \mathcal{L}_{i-\upsilon}^{x_i}) \mathcal{E}^m)$$

for $\mathfrak{g} = \mathfrak{sp}(\infty)$.
For $\mathfrak{g} = \mathfrak{sl}(\infty)$ $\mathrm{Ann}\mathcal{L}_1^\infty = \mathrm{Ann}\mathcal{R}_1^\infty$, so the notation $I(\upsilon, Q_f)$ makes sense.

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Theorem

The correspondence

$$Q \to I(\upsilon, Q_f)$$

is a bijection between the set of irreducible left cls and the set of prime integrable ideals in $U(\mathfrak{sl}(\infty))$*. The same correspondence is a bijection between the set of all irreducible cls and all prime ideals in* $U(\mathfrak{g})$ *for* $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$*.*

Fact: An integrable ideal of $U(\mathfrak{g})$ is prime if and only if it is primitive.

Open questions:

Conjecture: The augmentation ideal is the unique maximal ideal in $U(\mathfrak{sl}(\infty))$; $U(\mathfrak{o}(\infty))$ has two maximal ideals: the augmentation ideal and $I(\mathcal{R})$; $U(\mathfrak{sp}(\infty))$ has two maximal ideals: the augmentation ideal and the kernel of the homomorphism $U(\mathfrak{sp}(\infty)) \to W_\infty$.

Question: Are there non-integrable ideals in $U(\mathfrak{sl}(\infty)), U(\mathfrak{o}(\infty))$?

Question: What is a good analogue of Duflo's theorem?