What is a Donaldson-Thomas invariant?

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<u>Question:</u> What is DT-theory?

<u>Answer:</u> a new way to think about moduli problems for objects in certain (abelian) categories

What's wrong with the old way?

<u>3 Problems:</u>

A) loose information

B) not deformation invariant

C) no wall-crossing formula

To problem A: (loose information)

Example:

$$\mathrm{Mat}(d,d)/\mathrm{Gl}(d) \overset{\text{set}}{\simeq} \{\text{conjugacy classes}\}$$

$\mathrm{Gl}(d)$ acting by conjugation

conjugacy classes: not a nice space (e.g. a variety)

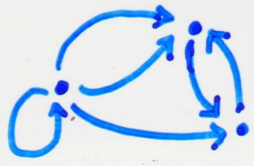
GIT quotient

$$X_d := \mathrm{Mat}(d,d) \xrightarrow{q} \mathrm{Mat}(d,d)/\!\!/\mathrm{Gl}(d) \simeq \mathbb{C}^d$$

$$A \longmapsto \begin{pmatrix}\text{eigenvalues of } A\\ \text{up to ordering}\end{pmatrix} = \left(\mathrm{tr}\,\Lambda^k A\right)_{k=1}^{d}$$

$$q^{-1}(\underline{0}) = \{\text{nilpotent } d\times d \text{ matrices}\} = \coprod_{\lambda\vdash d}\{\text{—"— } \mid \text{Jordan blocks of type } \lambda\}$$

$\Rightarrow$
- q identifies orbits
- $\mathbb{C}^d$ does not detect stabilizers

More general:  quiver (Q_0, Q_1) with relations

(fix a stab. cond.)

$d = (d_i)_{i \in Q_0}$

$$\underset{Q_1 \ni e: i \to j}{\prod} \mathrm{Mat}(d_j, d_i) \supset X_d^{ss} \xrightarrow{q} X_d^{ss} /\!\!/ G_d =: \mathcal{M}_d^{ss}$$

$$\circlearrowleft G_d = \prod_{i \in Q_0} Gl(d_i) \qquad \begin{matrix} \rotatebox{90}{$\in$} & \\ A & \longmapsto \mathrm{gr}\, A = \bigoplus_{k=1}^{\tau} S_k \end{matrix}$$

„polystable assoc. to A"

$\left(\exists \ 0 = A_0 \subset A_1 \subset \ldots \subset A_\tau = A \ , \ S_k := A_k / A_{k-1} \right.$ stable of the same slope $\left. \right)$

Jordan-Hölder type filtration

$\Rightarrow$
- q identifies all extensions of the S_k
- $\mathcal{M}_d^{ss}$ does not detect stabilizers

However: $$\mathcal{M}^{ss} := \coprod_{\theta(d) = \theta_0} \mathcal{M}_d^{ss} \supset \coprod_{\theta(d) = \theta_0} \mathcal{M}_d^{st} =: \mathcal{M}^{st}$$ $\leftarrow$ stable

satisfies $$\mathcal{M}^{ss} \text{ „=" } \mathrm{Sym}\, \mathcal{M}^{st} = \coprod_{\tau \geq 0} (\mathcal{M}^{st})^\tau / S_\tau$$

up to cut and paste

$$\Rightarrow \boxed{H_c^*(\mathcal{M}^{ss}, \mathbb{Q}) \text{ „=" } \mathrm{Sym}\, H_c^*(\mathcal{M}^{st}, \mathbb{Q})}$$

„master formula"

Solution to problem A according to DT-theory

Replace coh. of GIT quotient with equivariant coh.

$$H_c^*(\mathcal{M}^{ss},\mathbb{Q}) \rightsquigarrow \bigoplus_{\theta(d)=\theta_0} H^*_{G_d}(X_d^{ss},\mathbb{Q})^\vee[\text{shift}] =: H_c^*(\mathfrak{M}^{ss},\mathbb{Q})$$

$$H_c^*(\mathcal{M}^{st},\mathbb{Q}) \rightsquigarrow \bigoplus_{\theta(d)=\theta_0} H^*_{G_d}(X_d^{st},\mathbb{Q})^\vee[\text{shift}] =: H_c^*(\mathfrak{M}^{st},\mathbb{Q})$$

Facts: $H_c^*(\mathfrak{M}^{st},\mathbb{Q}) = H_c^*(\mathcal{M}^{st},\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z]z$, $\deg z = -2$

(Aut(stable) = $Gl(1)$)

$$H_c^*(\mathfrak{M}^{ss},\mathbb{Q}) \neq \operatorname{Sym} \underbrace{H_c^*(\mathfrak{M}^{st},\mathbb{Q})}$$ in general

need to correct this!

Define $V = \bigoplus_{\theta(d)=\theta_0} V_d^*$ by means of

$$\boxed{H_c^*(\mathfrak{M}^{ss},\mathbb{Q}) = \operatorname{Sym} V \otimes_{\mathbb{Q}} \mathbb{Q}[z]z}$$

Definition: (DT-inv., 1^{st} approach) $\widetilde{DT}_d = \sum_{m\in\mathbb{Z}} (-1)^m \dim V_d^m$

„replacement for $e(\mathcal{M}_d^{st})$"

Example: $\bigoplus_{d\geq 0} H^*_{Gl(d)}(\underbrace{Mat(d,d)}_{\sim \text{point}},\mathbb{Q})^\vee = \bigoplus_{d\geq 0} \operatorname{Sym}^d \mathbb{Q}[z]$

$$\Rightarrow V = V_1^* = \mathbb{Q}z^{-1} = H_c(\underbrace{\mathcal{M}^{st}}_{=\mathbb{C}},\mathbb{Q})$$

Interlude: equivariant cohomology

G algebraic group, e.g. $Gl(d)$

$\rightsquigarrow \exists$ $\begin{array}{c} EG \circlearrowleft_G \\ \downarrow \\ BG \end{array}$ (universal) G-principal bundle,

$\dim EG = \dim BG = \infty$, $\underset{\substack{\uparrow \\ \text{homotopy equivalent}}}{EG \sim \text{point}}$

e.g. $$\begin{array}{ccccc} EGl(d) = Hom^{inj}(\mathbb{C}^d, \mathbb{C}^\infty) & \ni & \varphi \\ \downarrow & & \downarrow \\ BGl(d) = Gr(d, \mathbb{C}^\infty) & \ni & \operatorname{im}\varphi \end{array}$$, e.g. $BGl(1) = \mathbb{P}^\infty$

Given a variety X with G-action $\rightsquigarrow$ $\begin{array}{c} EG \times_G X \\ \downarrow \\ BG \end{array}$ fiber $= X$

<u>Definition:</u> $H^*_G(X, \mathbb{Q}) := H^*(EG \times_G X, \mathbb{Q})$

e.g. $X \sim$ point $\circlearrowleft_{Gl(d)}$ $\Rightarrow$

$$\begin{aligned} H^*_{Gl(d)}(X, \mathbb{Q}) &= H^*(BGl(d), \mathbb{Q}) \\ &= H^*(Gr(d, \mathbb{C}^\infty), \mathbb{Q}) \\ &= \mathbb{Q}[c_1, \dots, c_d] \\ &= Sym^d \underbrace{\mathbb{Q}[c]}_{H^*_{Gl(1)}(pt, \mathbb{Q})}, \deg c = \end{aligned}$$

c_k = Chern classes of $EGl(d) \to BGl(d)$, $\deg c_k = 2k$

To problem B: (deformation invariance)

Example: consider $d \times d$ matrices A satisfying $A^n = 0$ for some fixed n

e.g. $d=1 \rightsquigarrow$ one solution $A = (0)$

small perturbation $A^n = \varepsilon \cdot \mathbb{1}$ (ε small)

e.g. $d=1 \rightsquigarrow$ n solutions $A = (\xi)$, $\xi \in \mu_n \cdot \sqrt[n]{\varepsilon}$

$\Rightarrow$ a deform. inv. sol'n should give $DT_{d=1} = n$

Solution according to DT-theory

write X_d^{ss} as a critical locus $\{\nabla f_d = 0\} = Crit(f_d)$ of a function $f_d : \underset{\circlearrowleft G_d}{U_d} \longrightarrow \mathbb{C}$, U_d smooth variety, f_d G_d-invariant

$\boxed{\text{Warning}}$ This works only for objects in a Calabi-Yau 3-category.
(e.g. relations are derivatives of a potential)

Example: $X_d = \{A \in Mat(d,d) \mid A^n = 0\} = Crit(f_d) \subset \underbrace{Mat(d,d)}_{U_d}$

$f_d(A) = tr\, A^{n+1}$

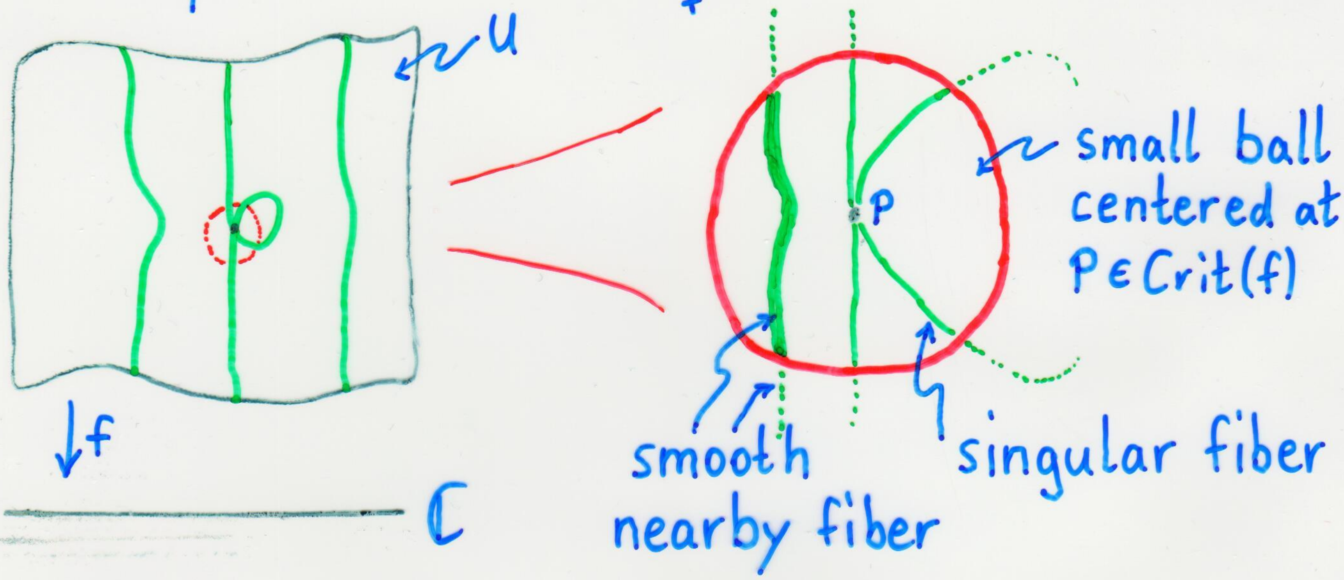
Now, replace $H^*_{G_d}(X_d^{ss}, \mathbb{Q})$ with $\mathbb{H}^*_{G_d}(X_d^{ss}, \phi_{f_d})$

ϕ_{f_d} = (pervers) sheaf of vanishing cycles of f_d

Interlude: sheaf of vanishing cycles

Given $U \xrightarrow{f} \mathbb{C}$, U smooth

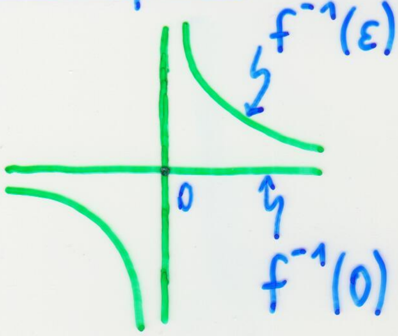
$\Rightarrow \exists$ perverse sheaf ϕ_f on $\mathrm{Crit}(f)$



$\underbrace{\text{Milnor fiber at } p}_{=:\, MF_f(p)}$:= smooth nearby fiber ∩ ball

The stalk $\phi_{f,p}$ of ϕ_f at $p \in \mathrm{Crit}(f)$ is the reduced cohomology $\mathring{H}^*(MF_f(p), \mathbb{Q})[\text{shift}]$.

Example: $f : \mathbb{C}^2 \ni (x,y) \longmapsto x \cdot y \in \mathbb{C}$



$MF_f(0) \sim S^1$

$\Rightarrow \phi_f = \phi_{f,0} = H^1(S^1, \mathbb{Q}) = \mathbb{Q}$

Solution to problem B

$$H^*_c(\mathfrak{M}^{ss},\mathbb{Q}) \rightsquigarrow \bigoplus_{\theta(d)=\theta_0} \mathbb{H}^*_{G_d}(X^{ss}_d, \phi_{f_d})^\vee[\text{shift}] =: \mathbb{H}^*_c(\mathfrak{M}^{ss}, \phi_f)$$

Define $W = \bigoplus_{\theta(d)=\theta_0} W^*_d$ by means of

$$\boxed{\mathbb{H}^*_c(\mathfrak{M}^{ss}, \phi_f) = \operatorname{Sym} W \otimes_{\mathbb{Q}} \mathbb{Q}[z]\sqrt{z}}$$

Definition: (DT-invariants, final version)

$$DT_d = \sum_{m\in\mathbb{Z}} (-1)^m \dim W^m_d$$

Example: matrices A satisfying $A^n = 0$

$$\Rightarrow \quad DT_d = \begin{cases} n & \text{for } d=1 \\ 0 & \text{else} \end{cases}$$

Open question: Is $W^*_d = H^*_c(Y_d, \mathbb{Q})$ for some variety Y_d?

may replace this by another canon. (perverse) sheaf

Thank you!